HW #9 Solution.

Problem 1

(a). $$\overline{T}_p(E) = \text{Trace} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E-\varepsilon+i\alpha & 0 \\ 0 & E-\varepsilon+i\beta \end{bmatrix}^{-1} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E-\varepsilon-i\alpha & 0 \\ 0 & E-\varepsilon-i\beta \end{bmatrix}^{-1}$$

$$= \text{Trace} \begin{bmatrix} \dfrac{\alpha^2}{(E-\varepsilon)^2+\alpha^2} & 0 \\ 0 & \dfrac{\beta^2}{(E-\varepsilon)^2+\beta^2} \end{bmatrix}$$

$$= \frac{\alpha^2}{(E-\varepsilon)^2+\alpha^2} + \frac{\beta^2}{(E-\varepsilon)^2+\beta^2}$$

(b) $$\overline{T}_A(E) = \text{Trace} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E-\varepsilon+\frac{i}{2}(\alpha+\beta) & 0 \\ 0 & E-\varepsilon+\frac{i}{2}(\alpha+\beta) \end{bmatrix} \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} E-\varepsilon-\frac{i}{2}(\alpha+\beta) & 0 \\ 0 & E-\varepsilon-\frac{i}{2}(\alpha+\beta) \end{bmatrix}$$

$$= \text{Trace} \begin{bmatrix} \dfrac{\alpha\beta}{(E-\varepsilon)^2+\left(\frac{\alpha+\beta}{2}\right)^2} & 0 \\ 0 & \dfrac{\alpha\beta}{(E-\varepsilon)^2+\left(\frac{\alpha+\beta}{2}\right)^2} \end{bmatrix}$$

$$= \frac{2\alpha\beta}{(E-\varepsilon)^2+\left(\frac{\alpha+\beta}{2}\right)^2}$$

(c)

$$\overline{T}_{PO} = \int_{-\infty}^{\infty} dE \left(\frac{\alpha^2}{(E-\varepsilon)^2+\alpha^2} + \frac{\beta^2}{(E-\varepsilon)^2+\beta^2} \right)$$

$$= \pi(\alpha+\beta) \qquad \left(\text{using } \int_{-\infty}^{\infty} \frac{\Gamma}{E^2+(\frac{\Gamma}{2})^2} dE = 2\pi \right)$$

$$\overline{T}_{AO} = \int_{-\infty}^{\infty} dE \, \frac{2\alpha\beta}{(E-\varepsilon)^2+\left(\frac{\alpha+\beta}{2}\right)^2}$$

$$= \pi \frac{4\alpha\beta}{\alpha+\beta}$$

$$MR = \frac{\overline{T}_{PO}-\overline{T}_{AO}}{\overline{T}_{PO}} = \frac{(\alpha+\beta) - \frac{4\alpha\beta}{\alpha+\beta}}{\alpha+\beta} = \left(\frac{\alpha-\beta}{\alpha+\beta} \right)^2$$

(d)

$$D(E) = \frac{1}{2\pi} \text{Trace } i \left[\begin{bmatrix} \frac{1}{E-\varepsilon+i\alpha} & 0 \\ 0 & \frac{1}{E-\varepsilon+i\beta} \end{bmatrix} - \begin{bmatrix} \frac{1}{E-\varepsilon-i\alpha} & 0 \\ 0 & \frac{1}{E-\varepsilon-i\beta} \end{bmatrix} \right]$$

$$= \frac{1}{2\pi} \text{Trace} \begin{bmatrix} \frac{2\alpha}{(E-\varepsilon)^2+\alpha^2} & 0 \\ 0 & \frac{2\beta}{(E-\varepsilon)^2+\beta^2} \end{bmatrix}$$

$$= \frac{1}{\pi} \left(\frac{\alpha}{(E-\varepsilon)^2+\alpha^2} + \frac{\beta}{(E-\varepsilon)^2+\beta^2} \right)$$

$$\int dE \; D(E) = 1 + 1 = 2$$

(e)

$$D(E) = \frac{1}{2\pi} \text{Trace } i \left[\begin{bmatrix} \frac{1}{E-\varepsilon+\frac{i}{2}(\alpha+\beta)} & 0 \\ 0 & \frac{1}{E-\varepsilon+\frac{i}{2}(\alpha+\beta)} \end{bmatrix} - \begin{bmatrix} \frac{1}{E-\varepsilon-\frac{i}{2}(\alpha+\beta)} & \\ & \frac{1}{E-\varepsilon-\frac{i}{2}(\alpha+\beta)} \end{bmatrix} \right]$$

$$= \frac{1}{\pi} \frac{\alpha+\beta}{(E-\varepsilon)^2+\left(\frac{\alpha+\beta}{2}\right)^2}$$

$$\int dE \; D(E) = 2$$

Problem 2

(a) $\hat{n}_\uparrow = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$, $\hat{n}_\downarrow = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$

Using basis transformation (from $\hat{n} \to \vec{z}$ basis).

$$\begin{array}{c|cc} & \hat{n}_\uparrow & \hat{n}_\downarrow \\ \hline z_\uparrow & \cos\theta/2 & -\sin\theta/2 \\ z_\downarrow & \sin\theta/2 & \cos\theta/2 \end{array} \quad \begin{array}{c|cc} & \hat{n}_\uparrow & \hat{n}_\downarrow \\ \hline \hat{n}_\uparrow & \alpha & 0 \\ \hat{n}_\downarrow & 0 & \beta \end{array} \quad \begin{array}{c|cc} & z_\uparrow & z_\downarrow \\ \hline \hat{n}_\uparrow & \cos\theta/2 & \sin\theta/2 \\ \hat{n}_\downarrow & -\sin\theta/2 & \cos\theta/2 \end{array}$$

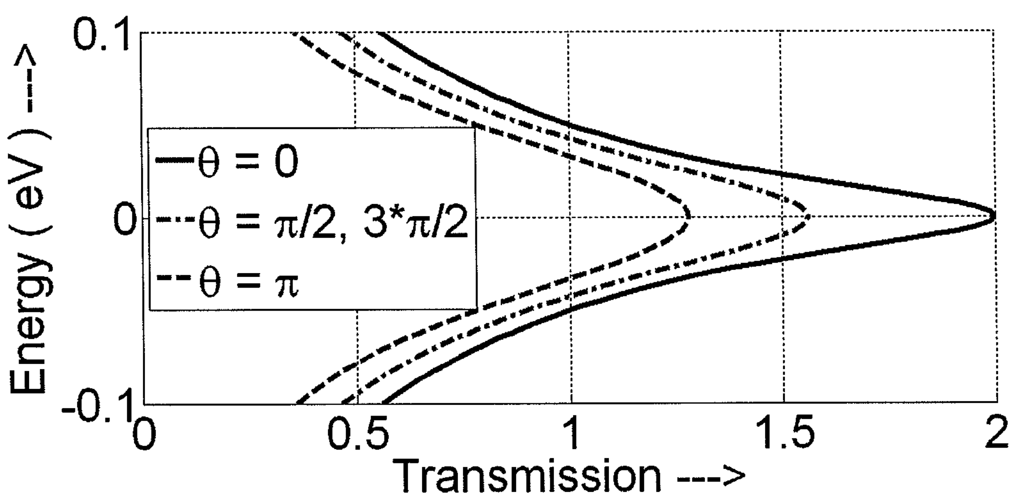
$$= \begin{array}{c} z_\uparrow \\ z_\downarrow \end{array} \begin{bmatrix} \alpha\cos^2\theta/2 + \beta\sin^2\theta/2 & (\alpha-\beta)\cos\theta/2\,\sin\theta/2 \\ (\alpha-\beta)\cos\theta/2\,\sin\theta/2 & \alpha\sin^2\theta/2 + \beta\cos^2\theta/2 \end{bmatrix}$$

(b)

```
% 2 levels transmission, changing analyzer
clear all
sx=[0 1;1 0];sy=[0 -i;i 0];sz=[1 0;0 -1];
g1=0.1;g2=0.1;Pc=0.25;zplus=i*1e-12;Np=1;

jj=1;
for theta=0:pi/2:pi
ii=1;
U=[cos(theta/2) -sin(theta/2);sin(theta/2) cos(theta/2)];
HB=0;dE=0.0025;for EE=-.1:dE:+.1
H=HB*sz;
    sig1=-i*0.5*g1*[1 0;0 Pc];sig2=-i*0.5*g2*[1 0;0 Pc];sig2=U*sig2*U';
    gam1=i*(sig1-sig1');gam2=i*(sig2-sig2');
    G=inv(((EE+zplus)*eye(2))-H-sig1-sig2);
    TM(ii,jj)=real(trace(gam1*G*gam2*G'));
    E(ii)=EE;ii=ii+1;
end
jj=jj+1;
end

hold on
h=plot(TM(:,1),E,'b');
set(h,'linewidth',[2.0])
h=plot(TM(:,2),E,'m');
h=plot(TM(:,3),E,'r');
set(gca,'Fontsize',[36])
xlabel(' Transmission ---> ')
ylabel(' Energy ( eV ) ---> ')
grid on
```



Problem 3

(a)

$$I = \frac{q}{\hbar}\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}(f_1 - f_2)$$

$$= \frac{q\Gamma}{2\hbar}\left[\frac{1}{1+e} - \frac{1}{1+e^2}\right] \simeq 1.1\times10^{-8}\ [A]$$

(b) Power delivered by the battery

$$P = IV = \frac{\mu_1-\mu_2}{q}\frac{q\Gamma}{2\hbar}\left[\frac{1}{1+e} - \frac{1}{1+e^2}\right]$$

$$= \frac{k_BT\cdot\Gamma}{2\hbar}\left[\frac{1}{1+e} - \frac{1}{1+e^2}\right] \simeq 2.8\times10^{-10}\ [W]$$

(c) heat removed from contact 1 per second.

$$I\cdot\frac{(\varepsilon-\mu_1)}{q} = I\cdot\frac{k_BT}{q} \simeq 2.8\times10^{-10}\ [W]$$

(d) heat dissipated in contact 2 per second.

$$I\cdot\frac{(\varepsilon-\mu_2)}{q} = I\cdot\frac{2k_BT}{q} \simeq 5.6\times10^{-10}\ [W]$$

(e) heat removed from contact 1 (ignoring contact 2).

$$P[W] = I\cdot\frac{(\varepsilon-\mu_1)}{q} = \frac{\Gamma}{2\hbar}(\varepsilon-\mu_1)\frac{1}{1+e^{\left(\frac{E-\mu_1}{k_BT}\right)}}$$

$$P\left(\frac{\varepsilon-\mu_1}{k_BT}\right) \Rightarrow \quad P(x) = \text{constant} \times \frac{x}{1+e^x}$$

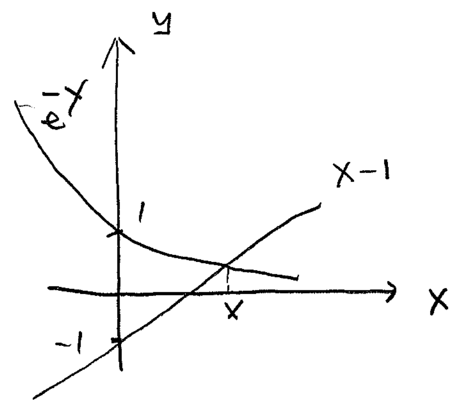
We need to find the extremal of $P(x)$

$$P'(x) = \frac{1}{1+e^x} + x\,\frac{(-1)\,e^x}{(1+e^x)^2}$$

$$= \frac{1+e^x - xe^x}{(1+e^x)^2} = 0$$

$$e^{-x} + 1 = x$$

$$e^{-x} = x - 1$$



$$x \simeq 1.3$$

$$\therefore\ E - \mu_1 \cong 1.3\, k_B T \simeq 0.03\ eV$$