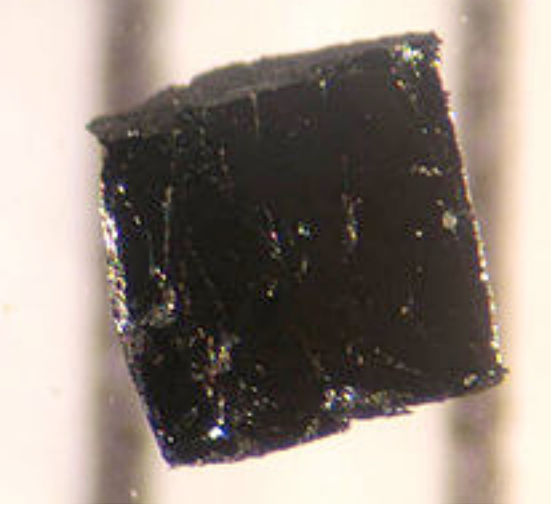
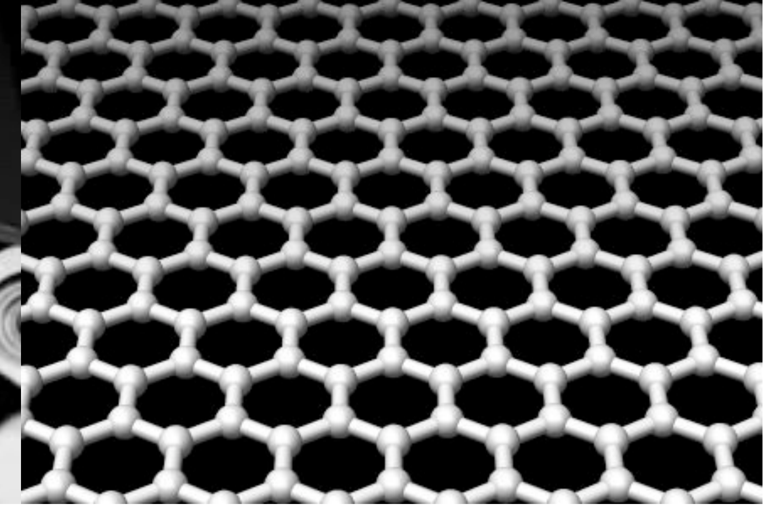
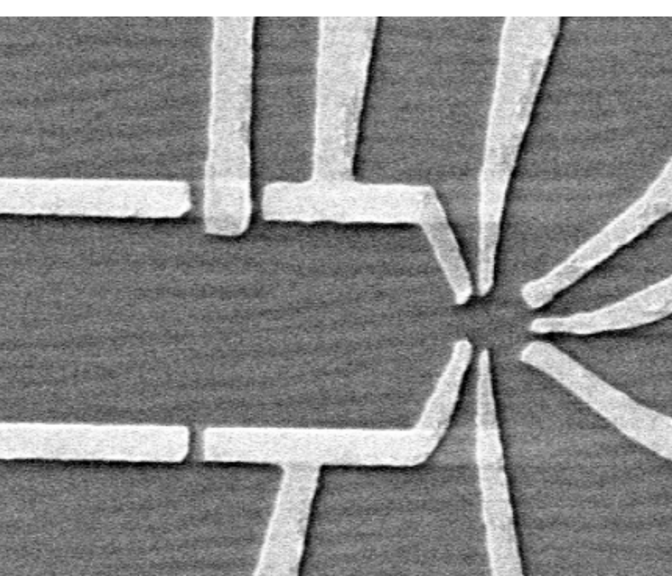




Modeling in Physics

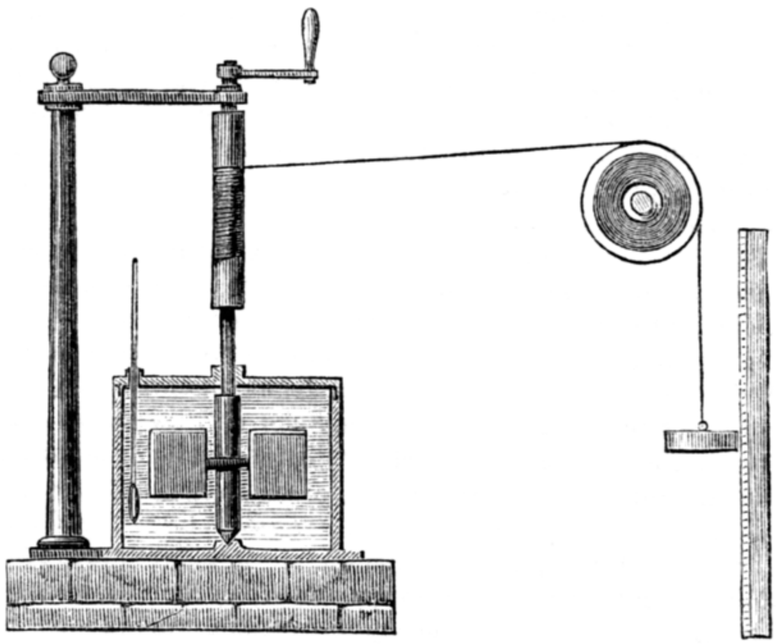
Bruno Uchoa
Department of Physics and Astronomy
University of Oklahoma

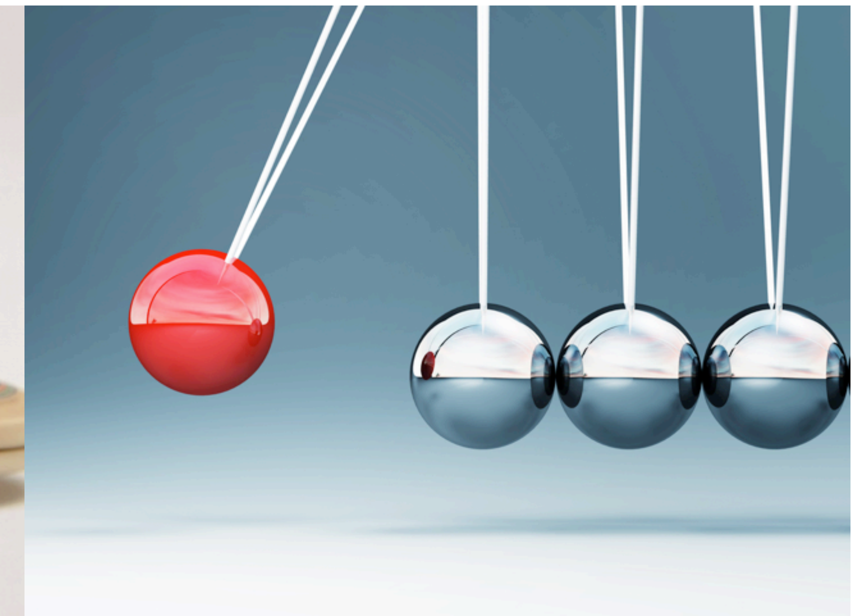


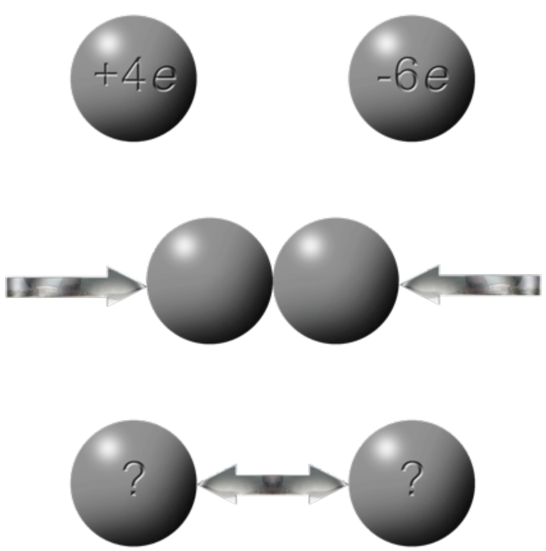

Lecture 3

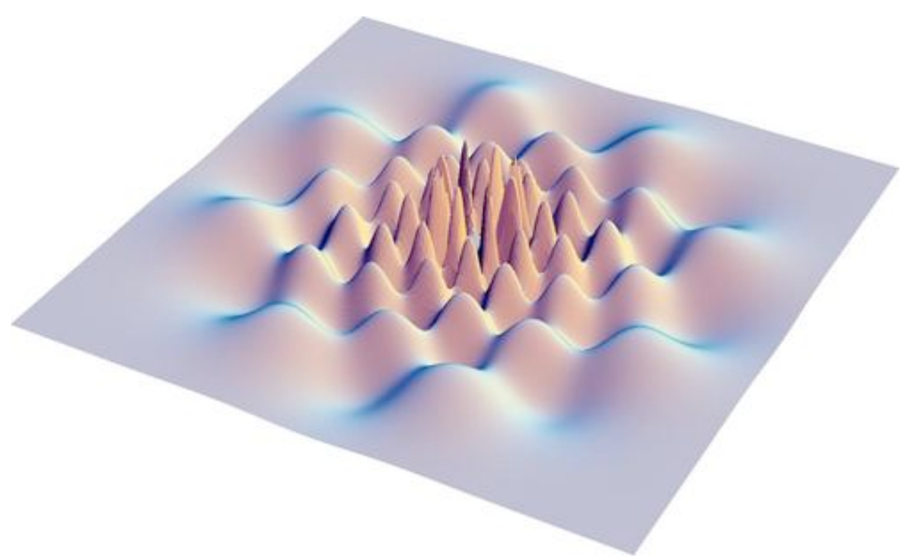
Conservation laws and broken symmetries

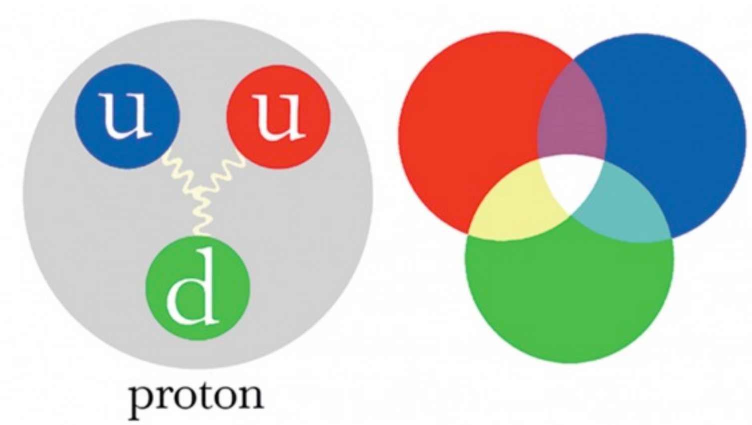
In nature, there is a set of physical quantities that are exactly conserved in every isolated system.











Symmetries in nature

Noether’s theorem

In 1915, Emmy Noether realized that every continuous symmetry corresponds to a conservation law.

Noether's theorem

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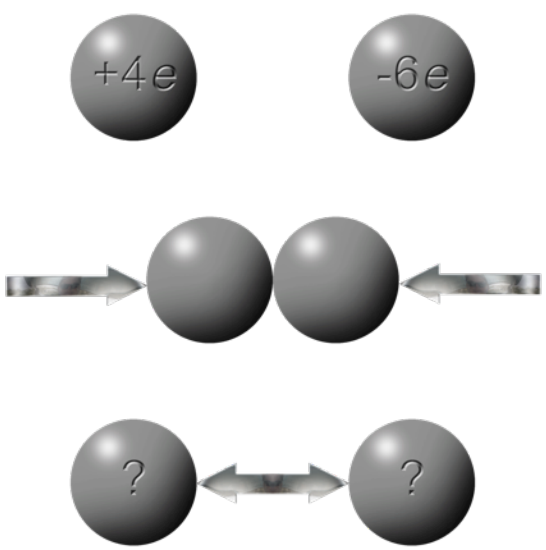


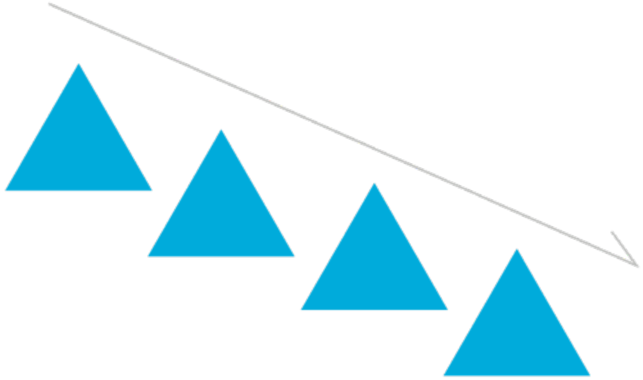
Conservation of charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Continuity equation

Noether's theorem

In 1915, Emmy Noether realized that every continuous symmetry corresponds to a conservation law.



Translational symmetry

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0$$

Conservation of momentum

Noether's theorem

In 1915, Emmy Noether realized that every continuous symmetry corresponds to a conservation law.

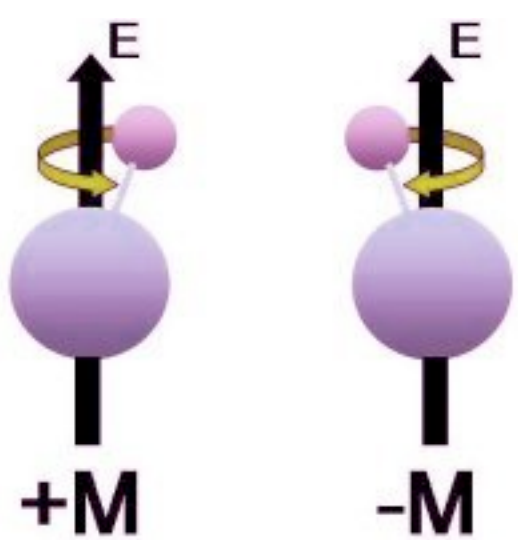
Rotational symmetry

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = 0$$

Conservation of angular momentum

Noether's theorem

In 1915, Emmy Noether realized that every continuous symmetry corresponds to a conservation law.



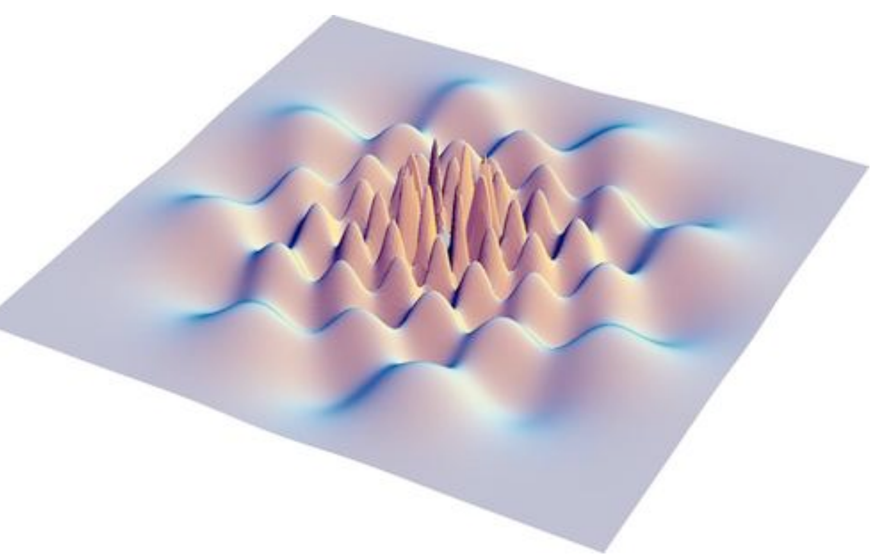
Time reversal symmetry

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

Conservation of energy

Noether's theorem

In 1915, Emmy Noether realized that every continuous symmetry corresponds to a conservation law.



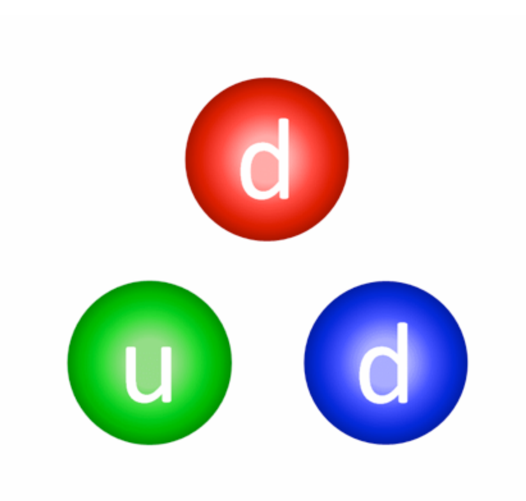
Conservation of probability

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

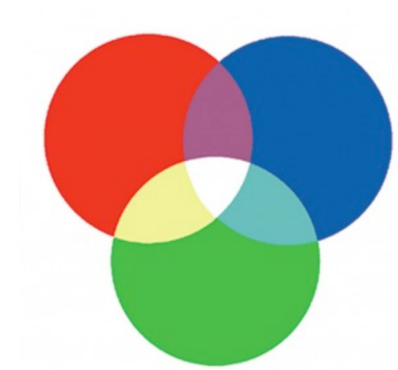
Continuity equation

Noether's theorem

In 1915, Emmy Noether realized that every continuous symmetry corresponds to a conservation law.



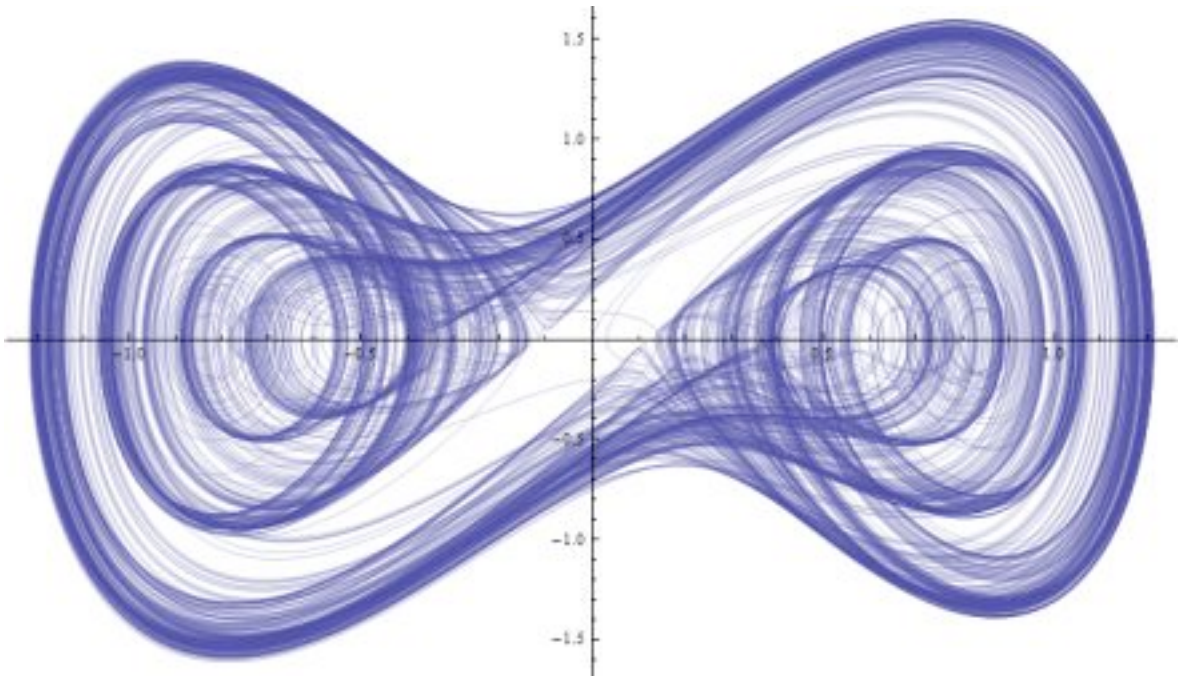
Conservation of color charge

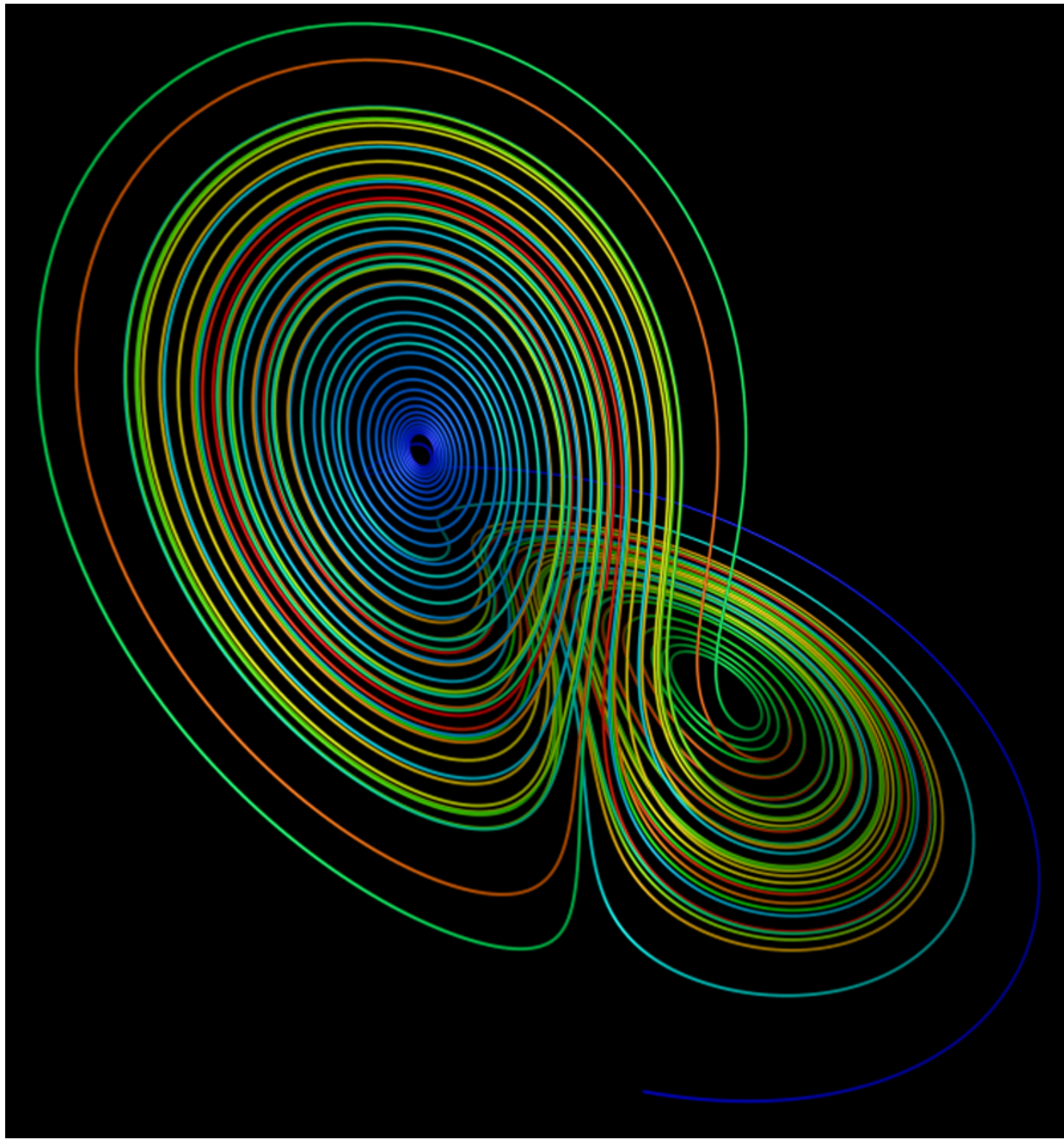


SU(3) gauge fields (gluons)

Other conservation laws are dynamically generated in nature.

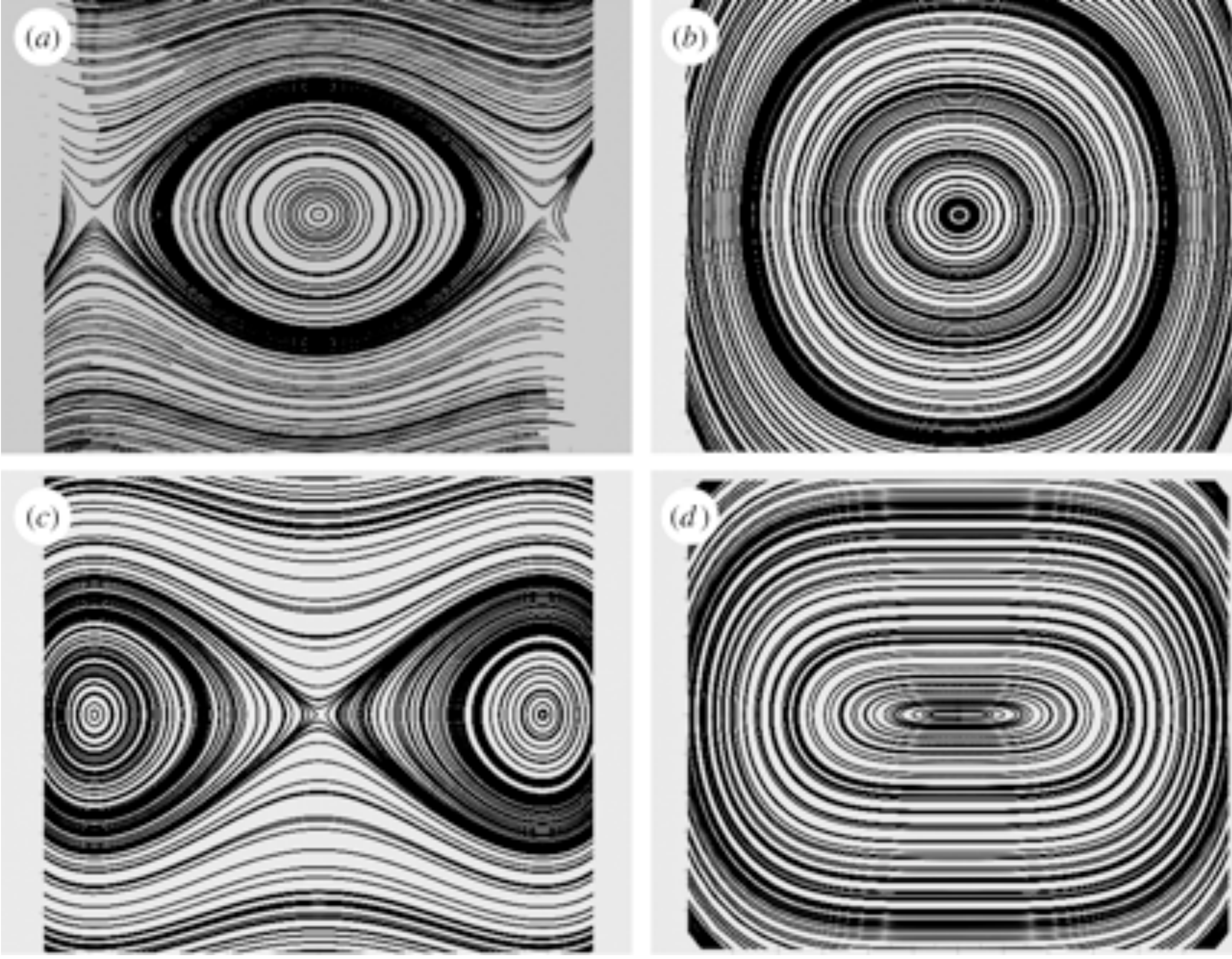
Phase space



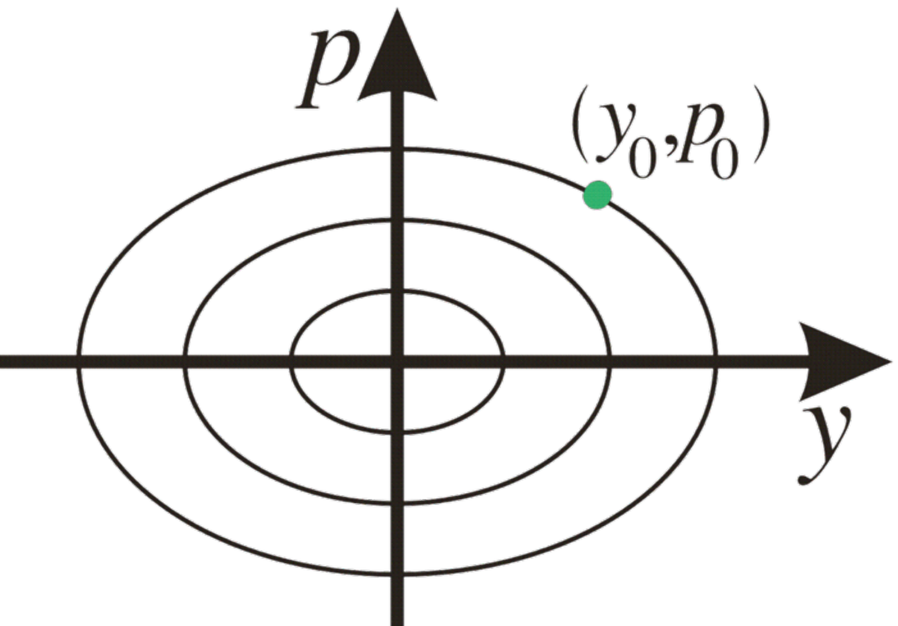


In nature, the universe of possible dynamic configurations of an isolated system can be mapped into semiclassical orbits inside the phase space.

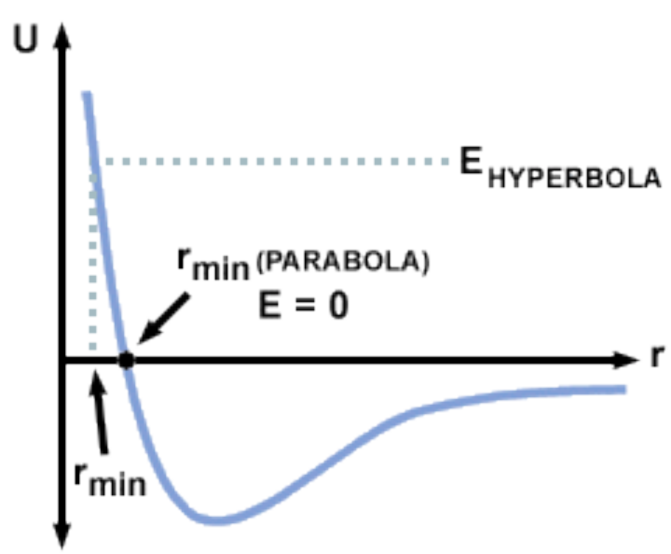
In a isolated systems, orbits can be either open or closed, regular or chaotic.

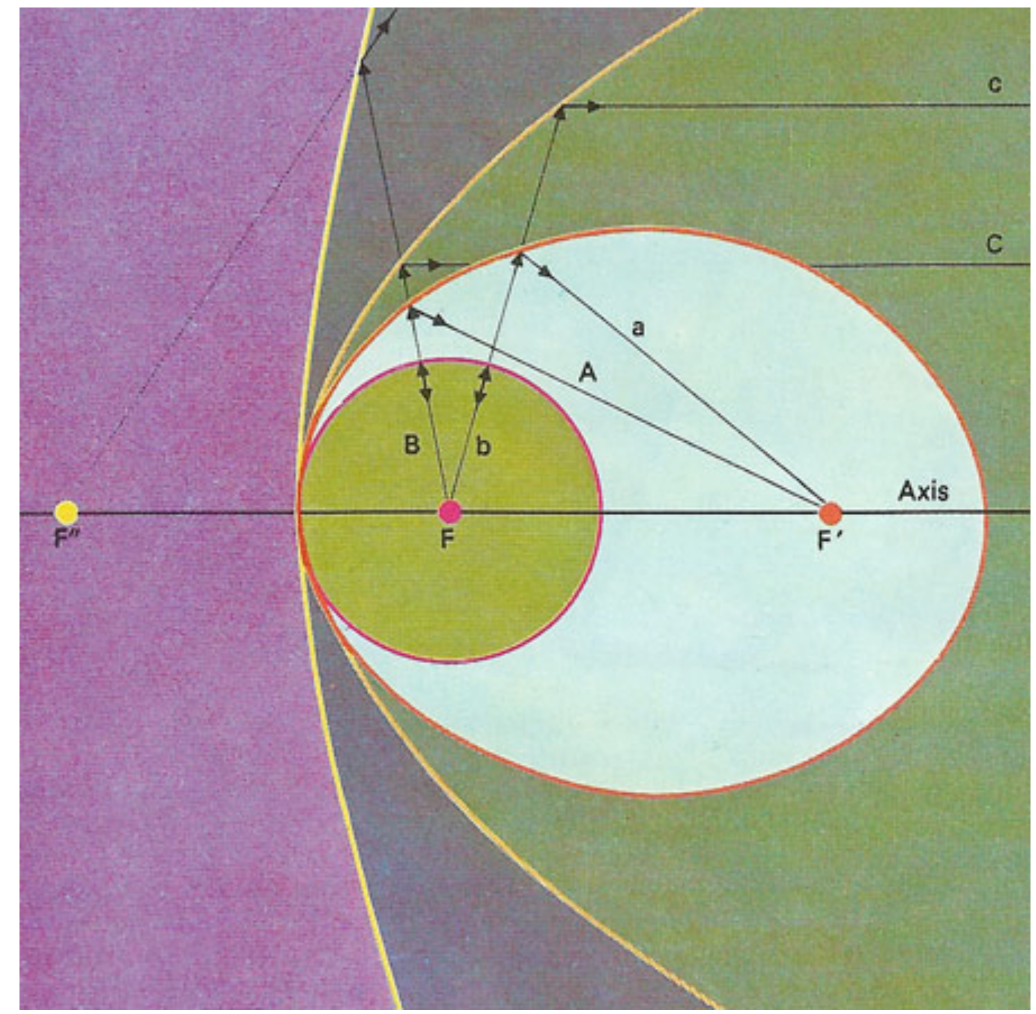


A 1D harmonic oscillator has two classical turning points and hence closed periodic orbits in the phase space velocity vs position (y)

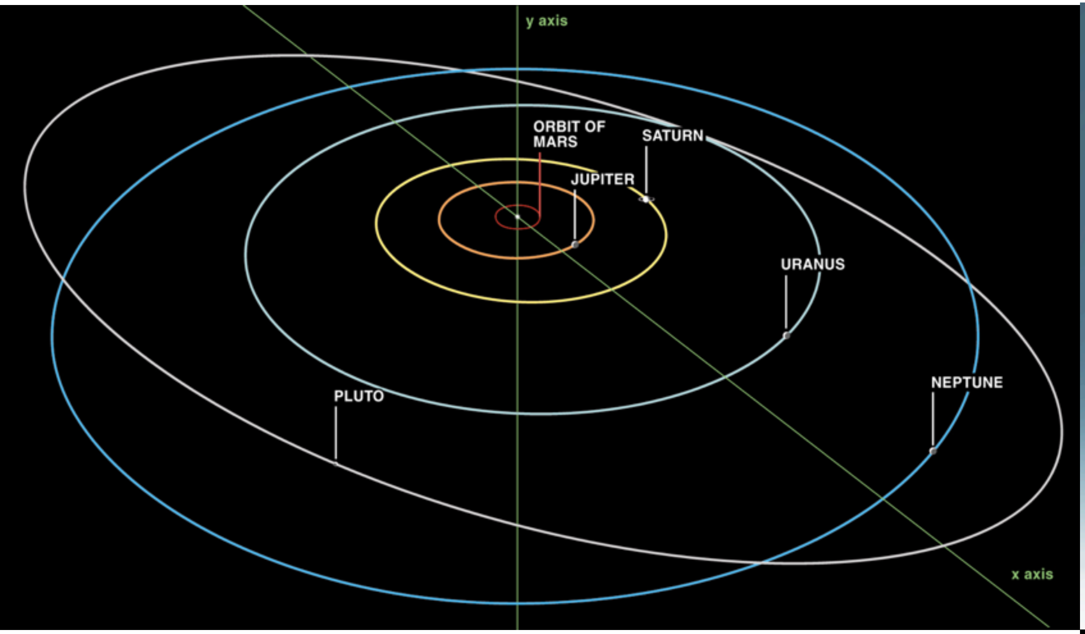


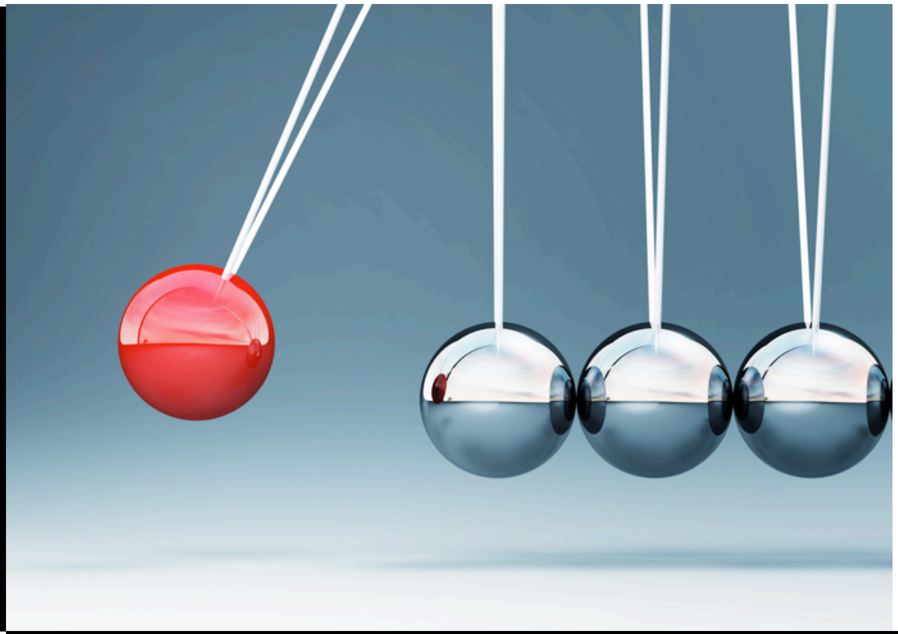
Gravitating objects with open orbits have a single classical turning point and explore an infinite phase space.

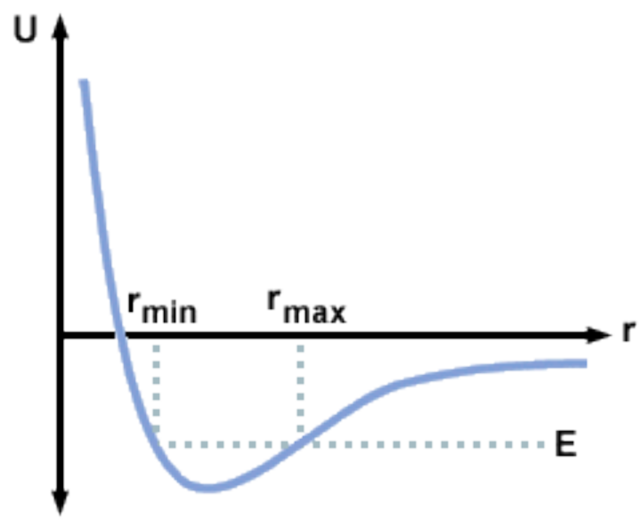


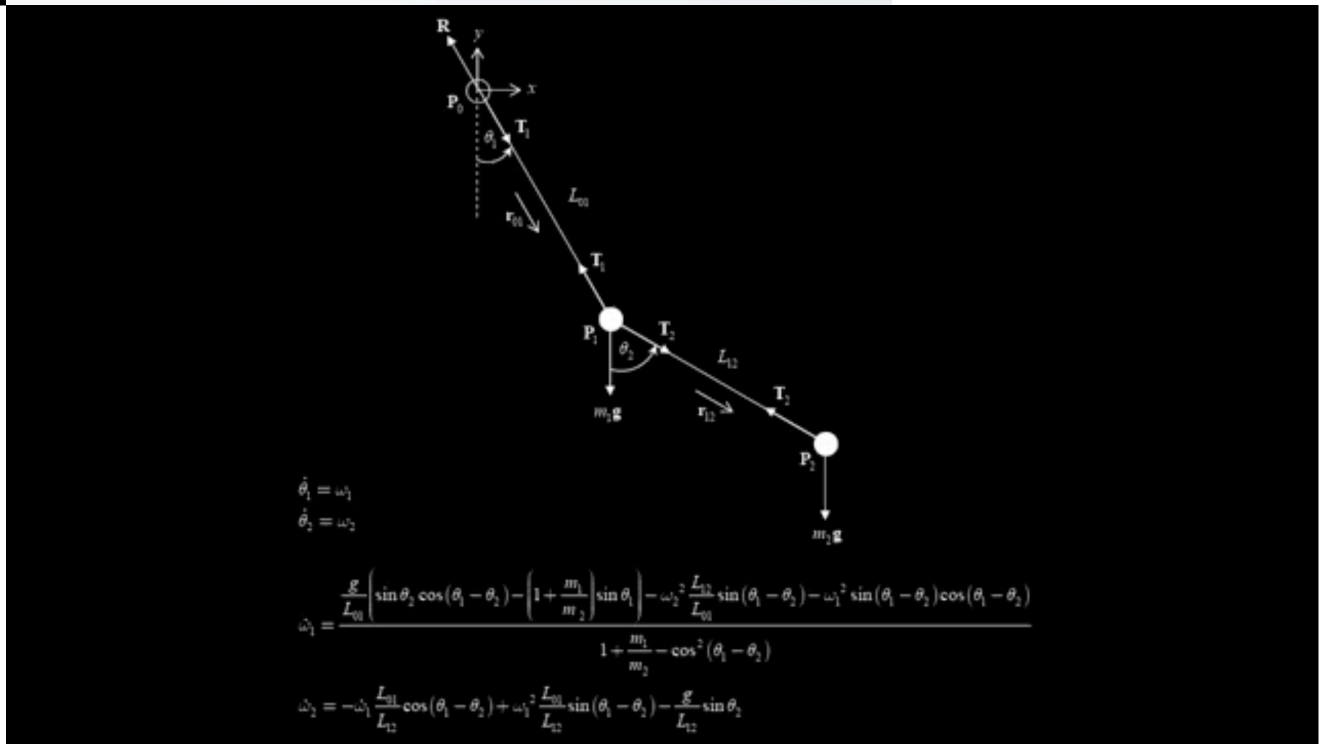


Complex isolated systems that explore a finite phase space must conserve time averages of certain observables





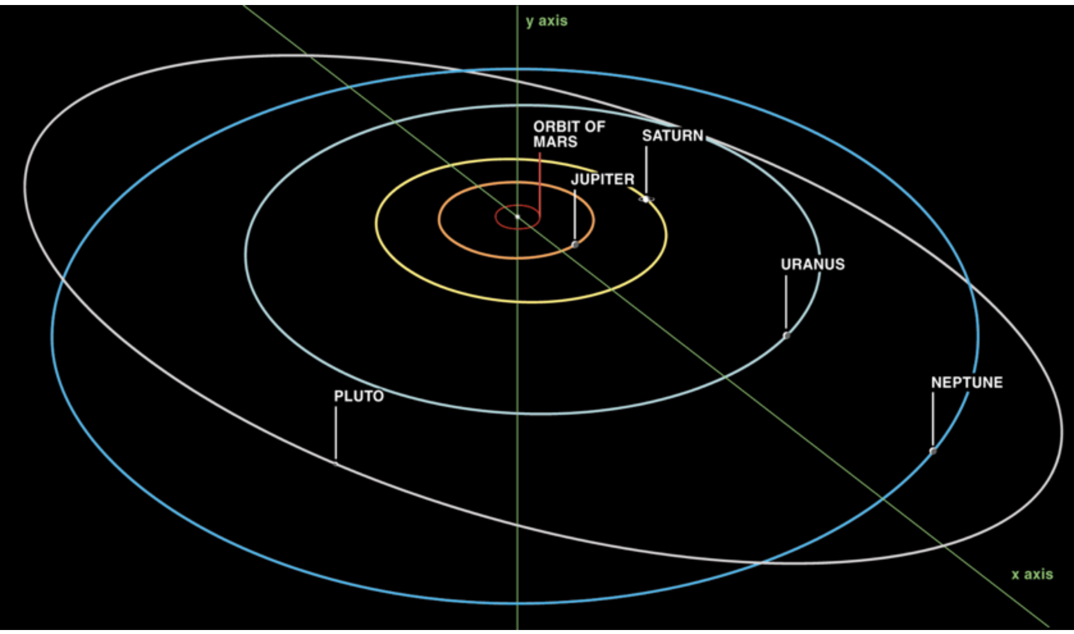


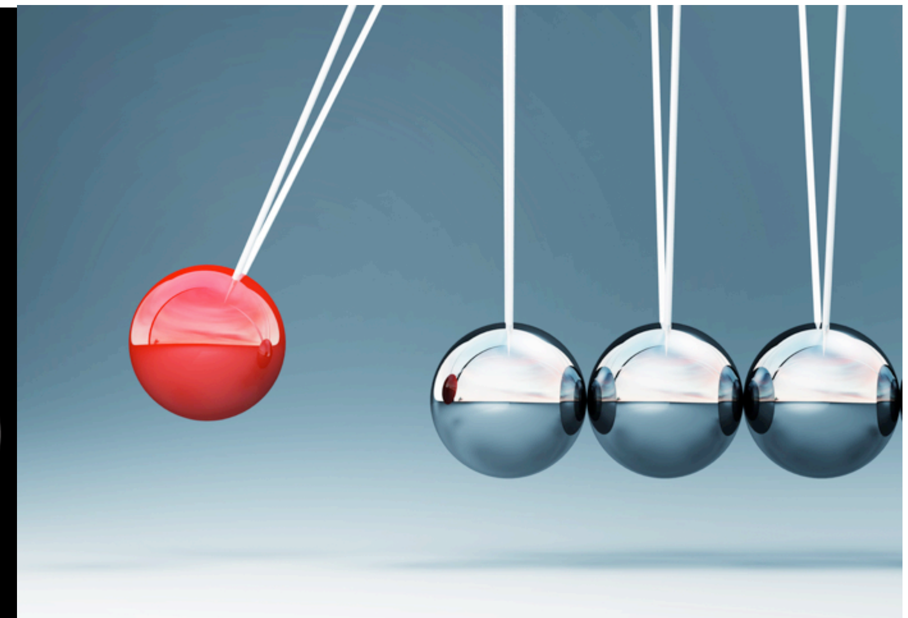


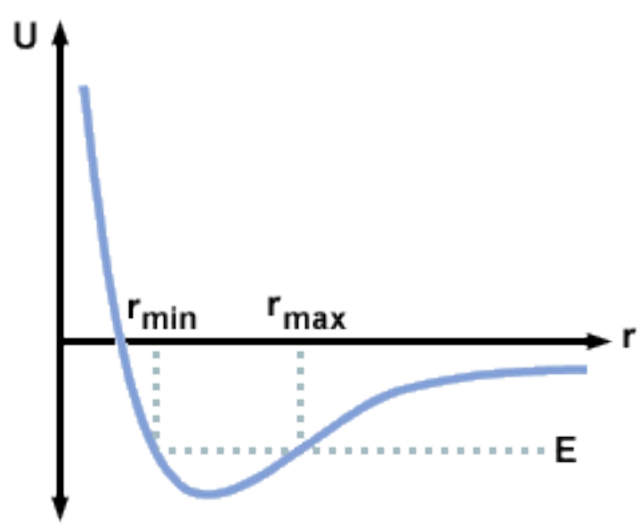
Closed periodic orbit

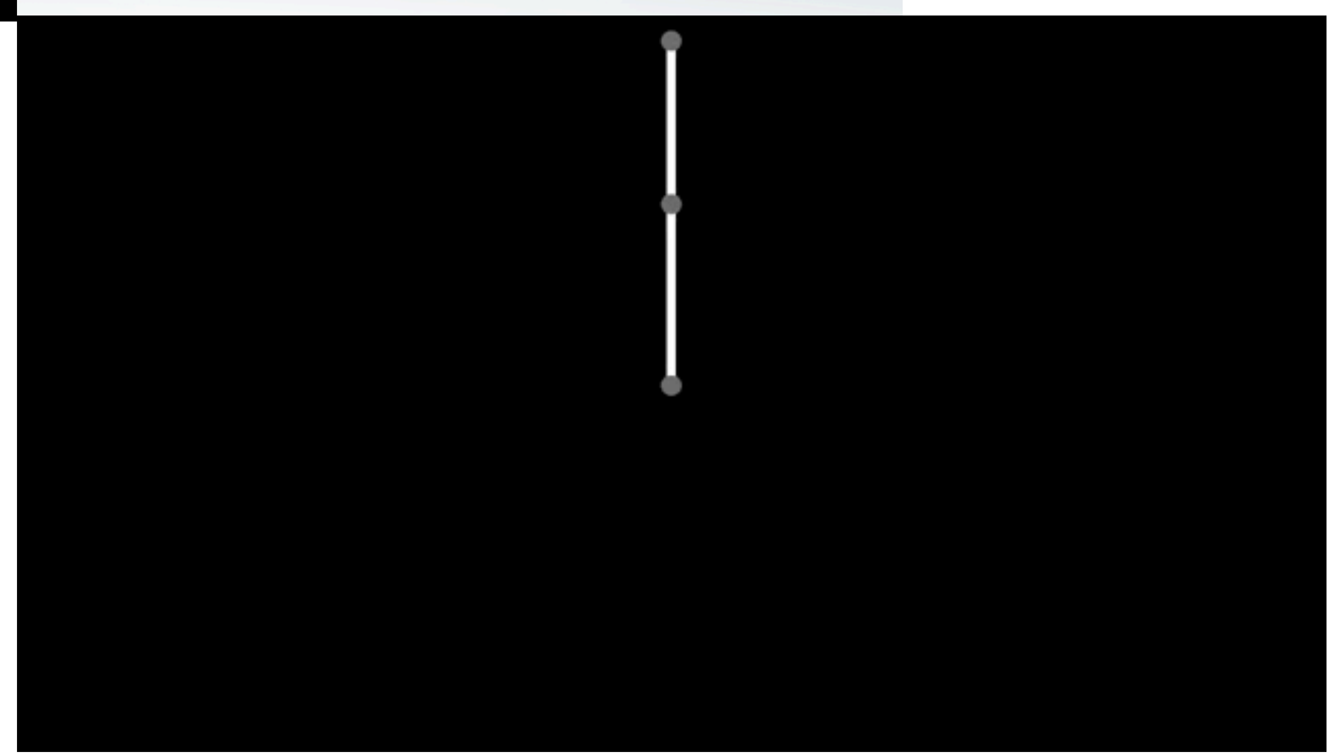
Open chaotic orbit

Complex isolated systems that explore a finite phase space must conserve time averages of certain observables





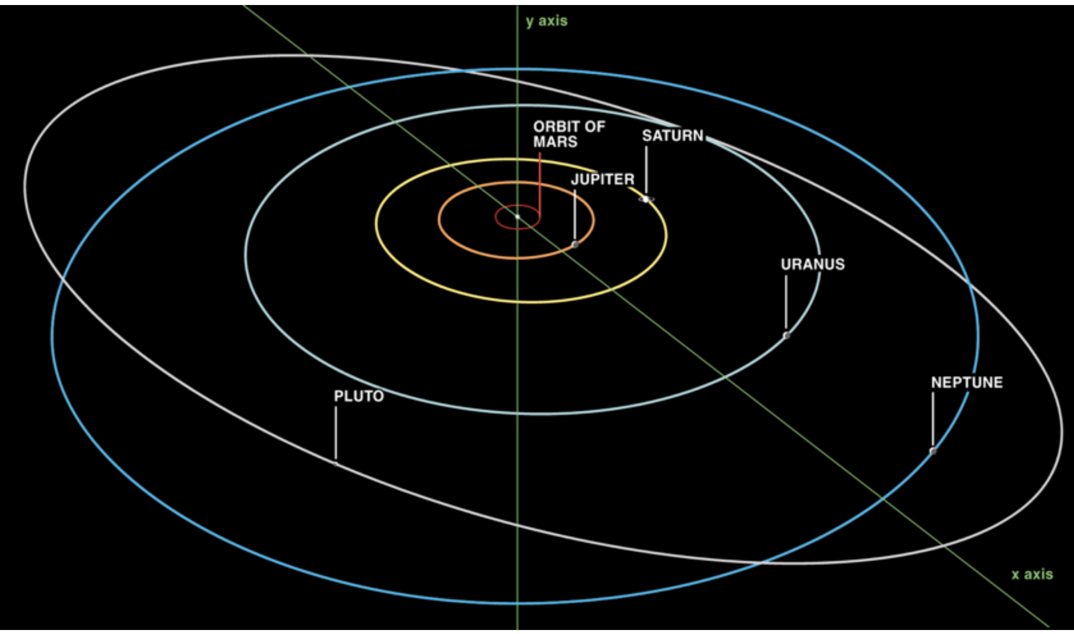


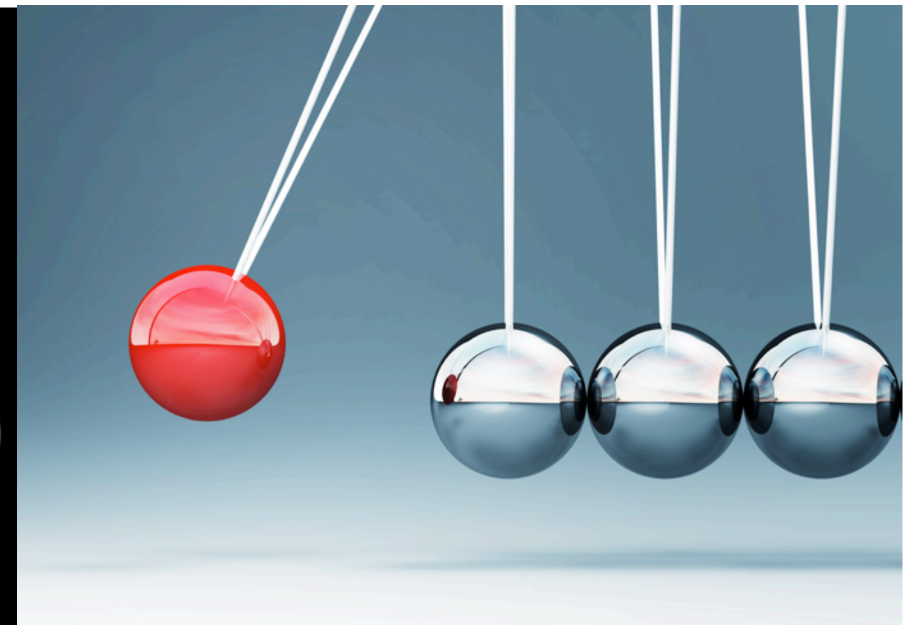


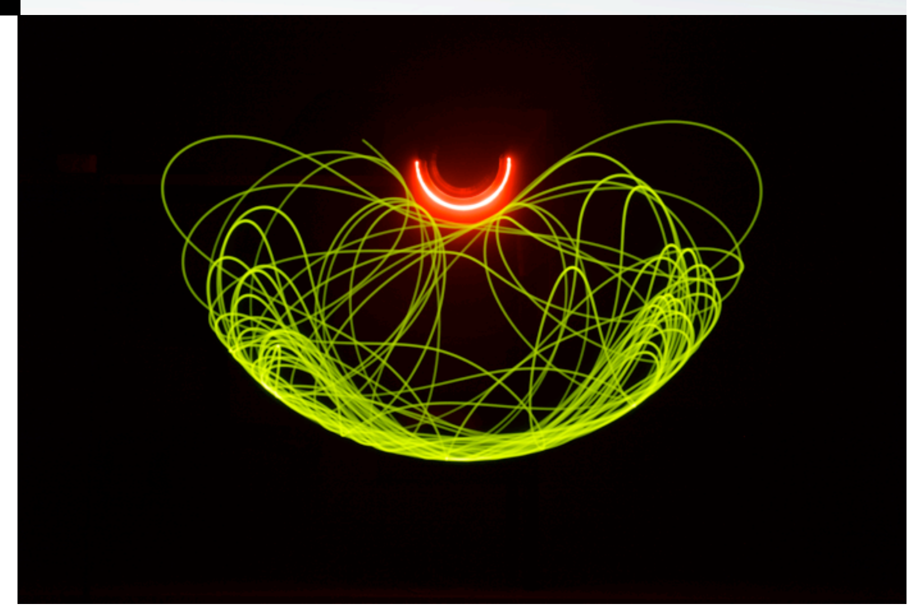
Closed periodic orbit

Open chaotic orbit

Complex isolated systems that explore a finite phase space must conserve time averages of certain observables







Even open chaotic orbits can be confined into a finite phase space.

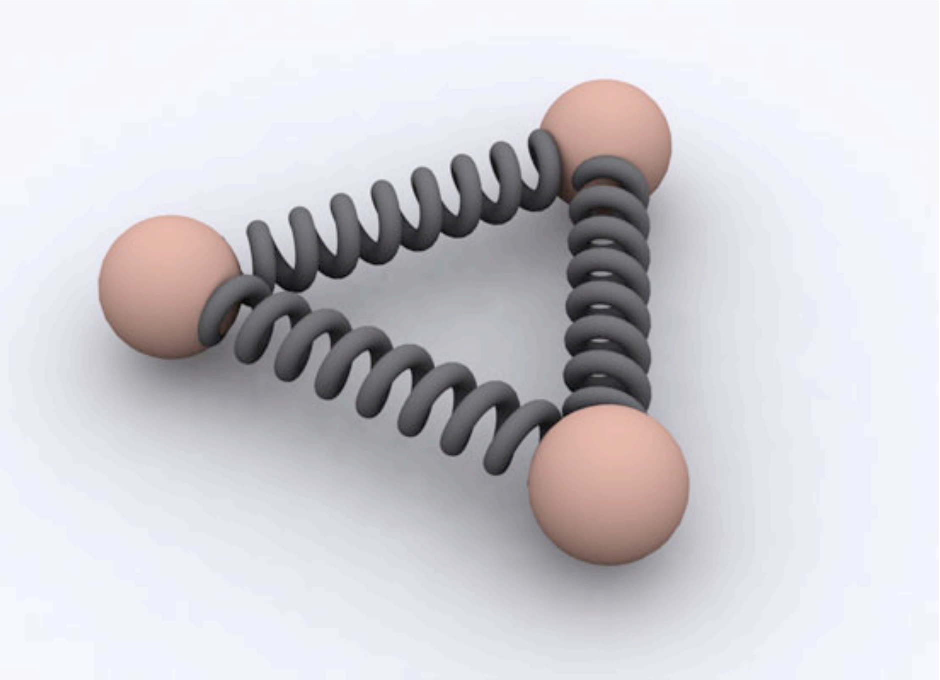
Virial theorem

If an isolated system with orbits confined to a finite phase space has a potential energy that is a homogeneous function of degree N, namely

$$V(\lambda x_1, \ldots, \lambda x_{3m}) = \lambda^N V(x_1, \ldots, x_{3m})$$

then the time average of the Kinetic energy is related to the time average of the potential energy by

$$2\langle K \rangle = N \langle V \rangle$$



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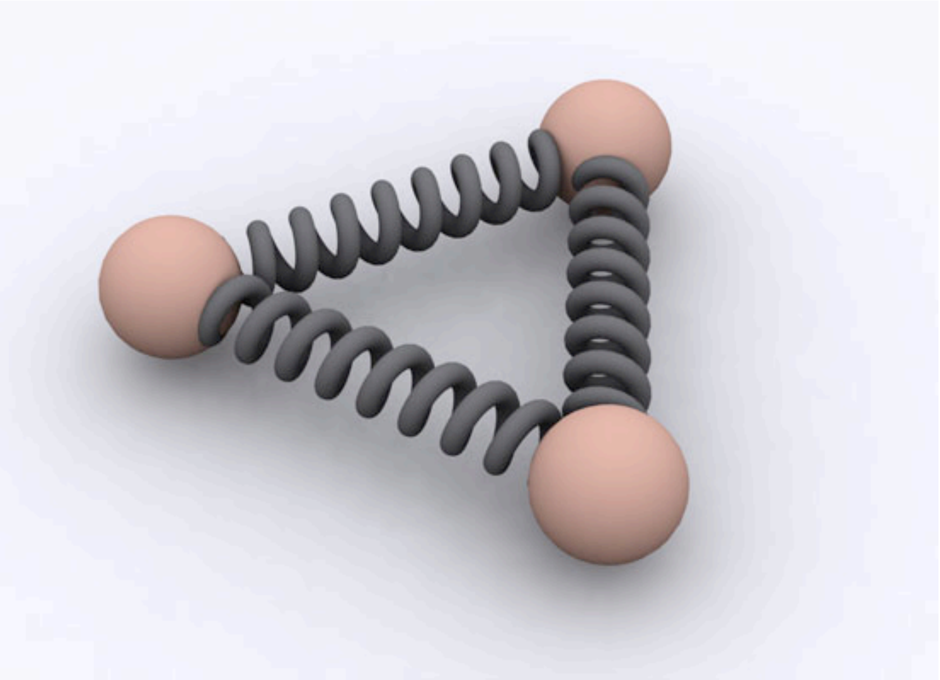

$$V(x_1, \ldots, x_m) \propto \sum_{i \neq j}^{m} |\mathbf{x}_i - \mathbf{x}_j|^2 \quad (N = 2)$$

$$E = \langle K \rangle + \langle K \rangle = 2\langle K \rangle$$

Virial theorem

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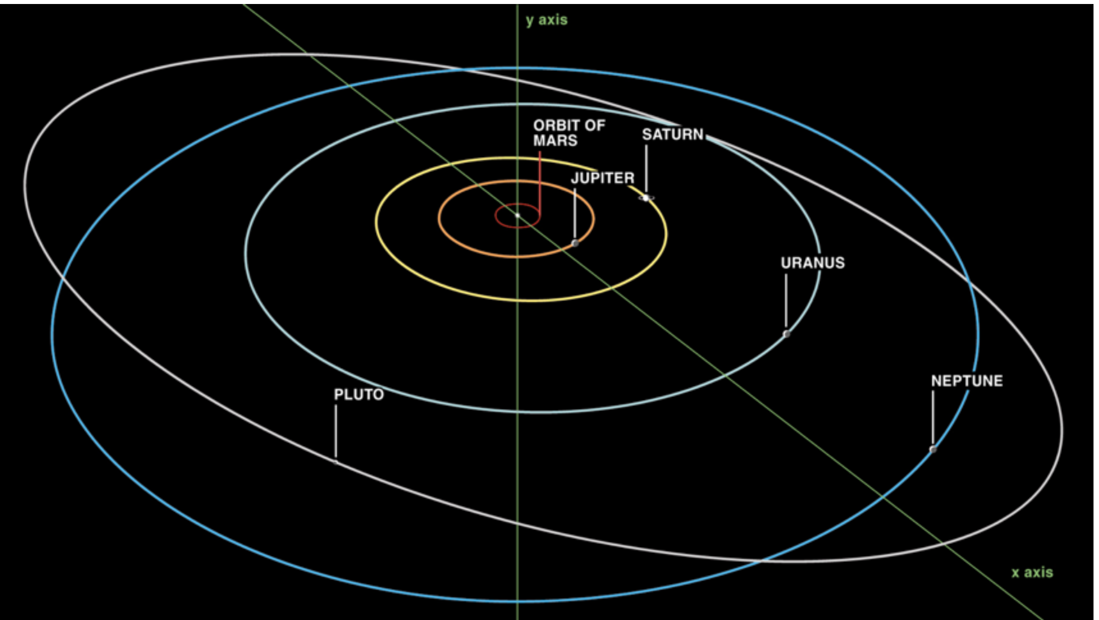

$$V(x_1, \ldots, x_m) \propto \sum_{i \neq j}^{m} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}$$

Virial theorem

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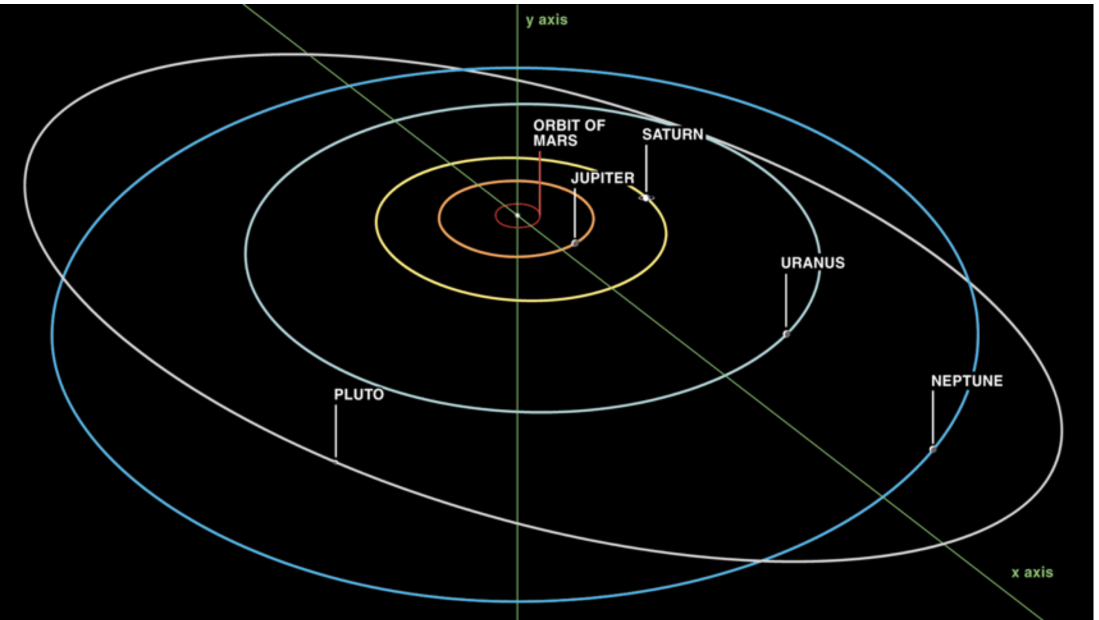


$$V(x_1, \ldots, x_m) \propto \sum_{i \neq j}^{m} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} \quad (N = -1)$$

$$E = \langle K \rangle - 2\langle K \rangle = -\langle K \rangle$$

In 1933, Fritz Zwicky studied the velocity of galaxies in the Coma cluster and estimated the mass based on the luminosity and number of galaxies inside the cluster.

A galaxy cluster can be thought of an an isolated system bound by internal gravitational forces. The phase space explored by the galaxies inside the cluster is finite.

Virial theorem

By measuring a lower bound of the relative velocities of galaxies inside clusters, one can calculate a lower bound of their masses through the Virial theorem,

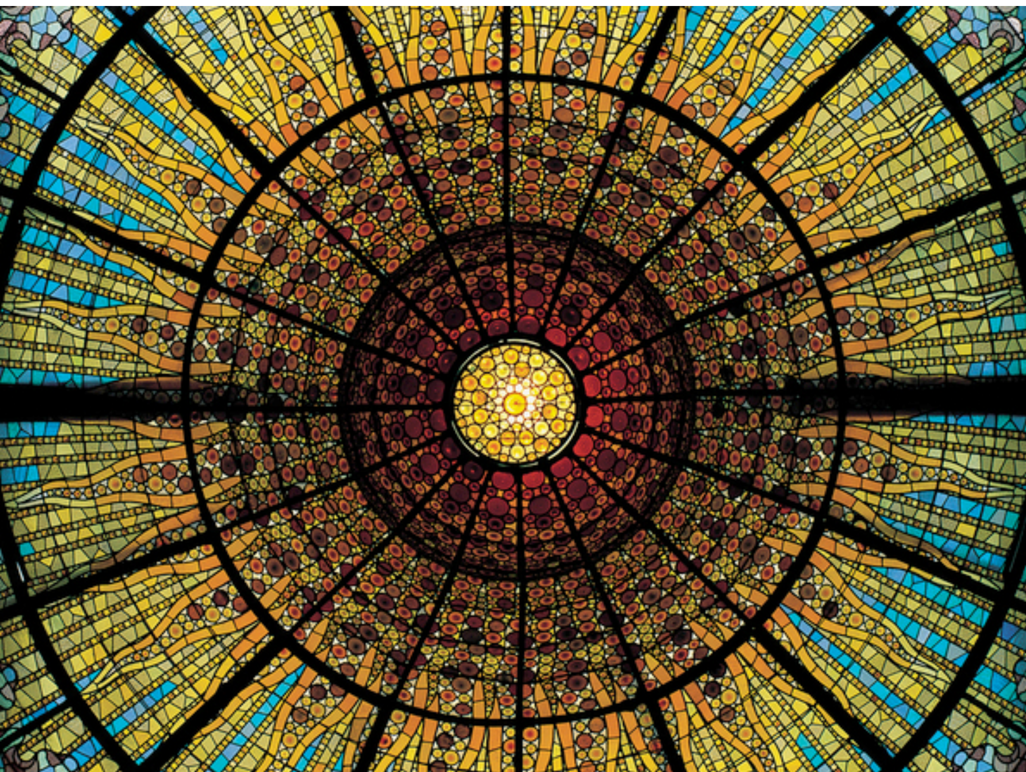
$$2\langle K\rangle = -\langle V\rangle$$

The currently observed masses (visible) are a lot smaller than the lower bound set by the Virial theorem. That implies in the existence of dark matter!

The validity of the Virial theorem and the existence of dark matter have been confirmed through gravitational lensing measurements, that can detect the total mass of galaxy clusters (visible and dark).

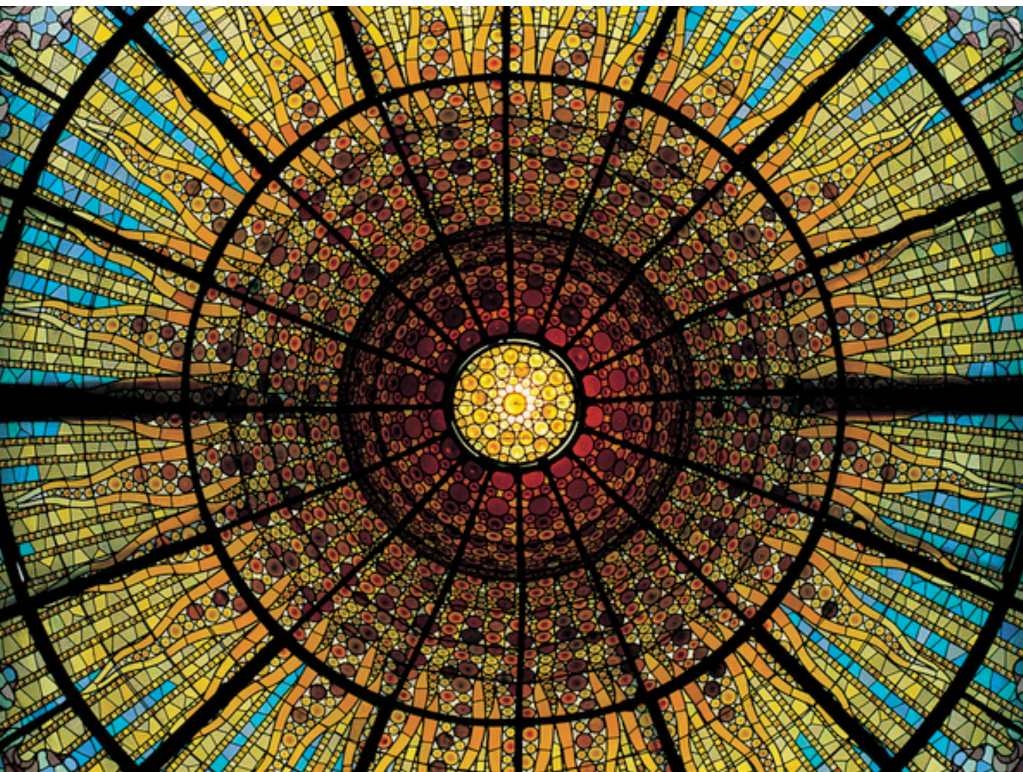
The physical nature and origin of dark matter remains an outstanding open puzzle!

There is another reason why we care about symmetry.



There is another reason why we care about symmetry.

Broken symmetries can tell us a lot about the underlaying physics.

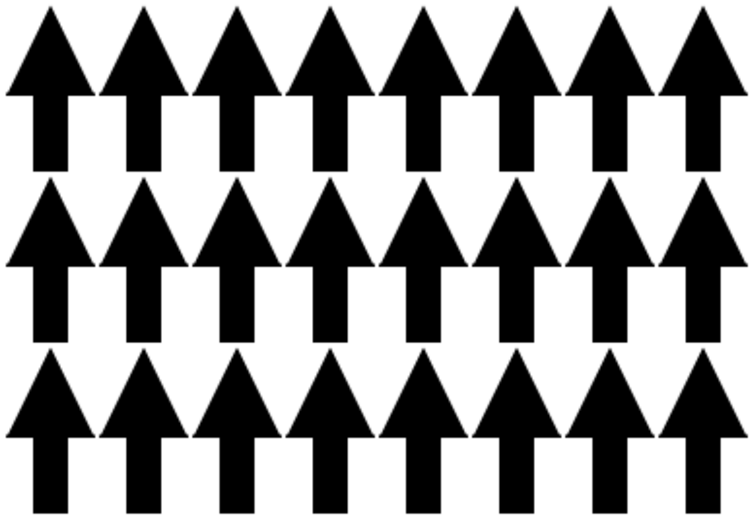


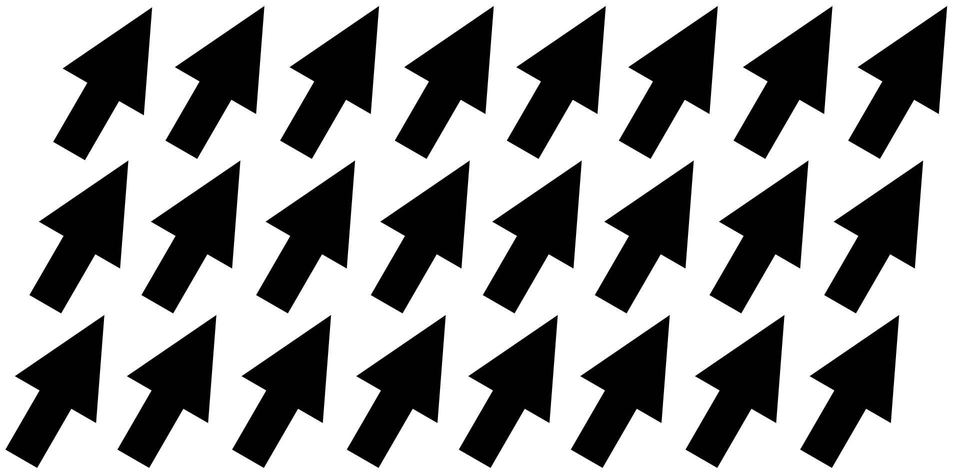
Spontaneously broken symmetries

Heisenberg spin exchange Hamiltonian

$$\mathcal{H} = J \sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$$

Spin on site i





The ground state of a quantum ferromagnet is degenerate under rotation, reflecting the symmetry of the Hamiltonian

Spontaneously broken symmetries

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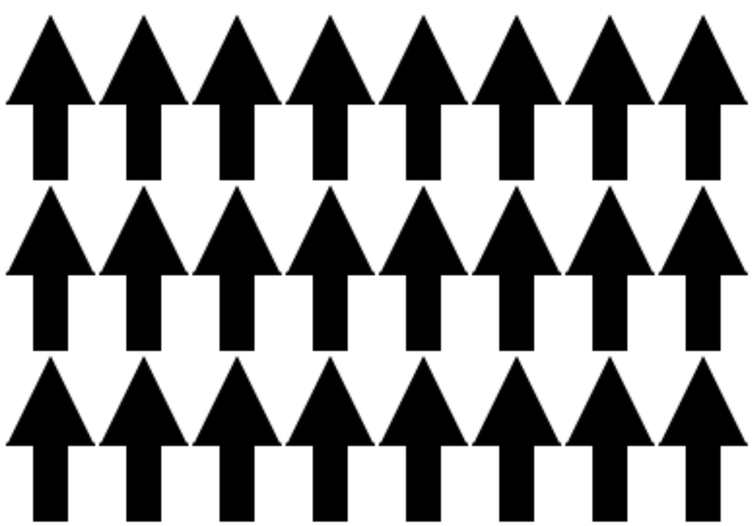

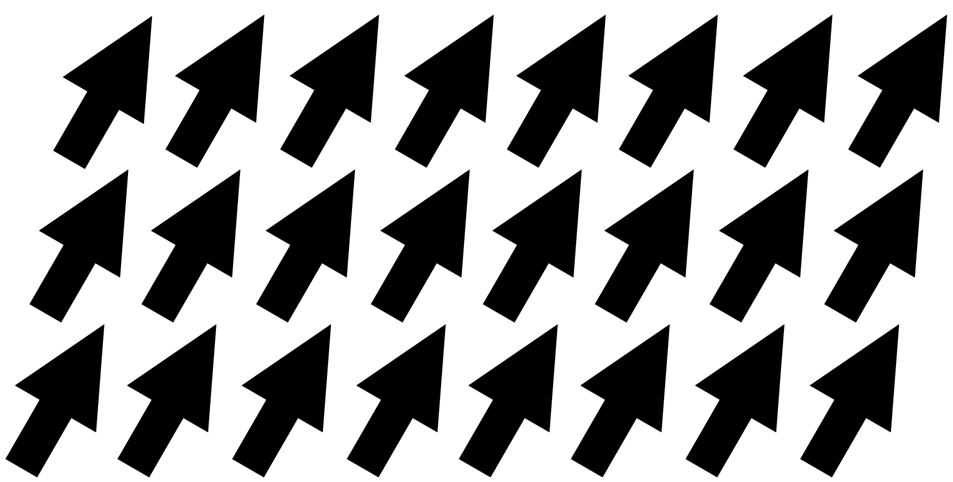

But once the system orders, the ground state chooses an arbitrary orientation, spontaneously lowering the original symmetry of the Hamiltonian.

Goldstone theorem

In 1961, J. Goldstone showed that the spontaneous breaking of any continuous symmetry is associated with a massless and spinless particle known as the Nambu-Goldstone boson.

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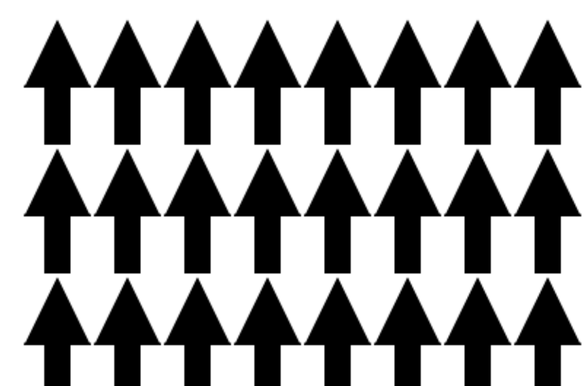
The ordered spins in a ferromagnet break rotational symmetry.



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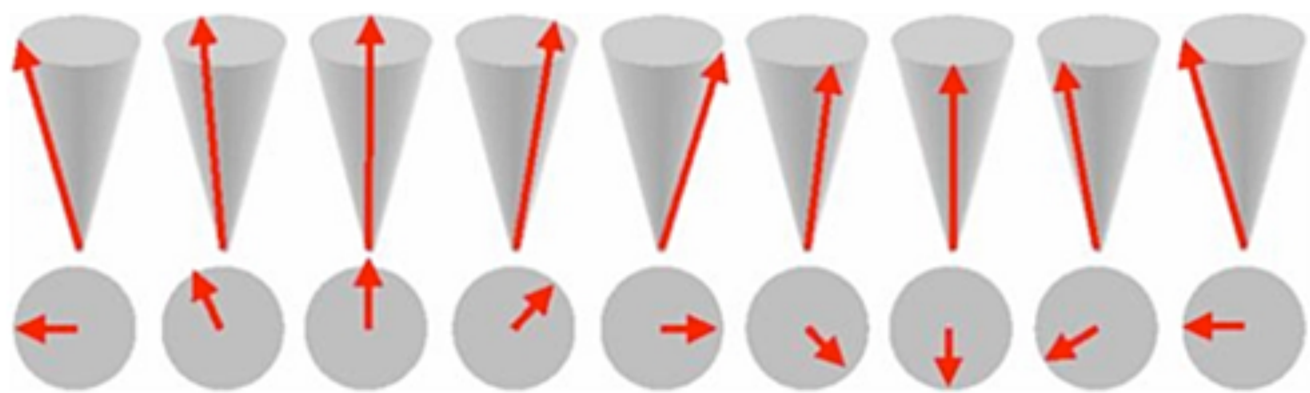
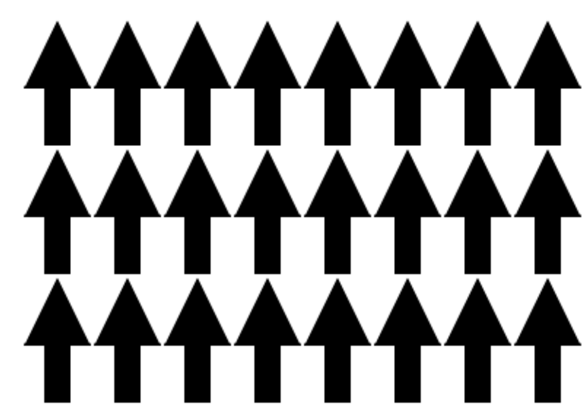


The elementary excitations of a quantum ferromagnet are spin waves (magnons)!



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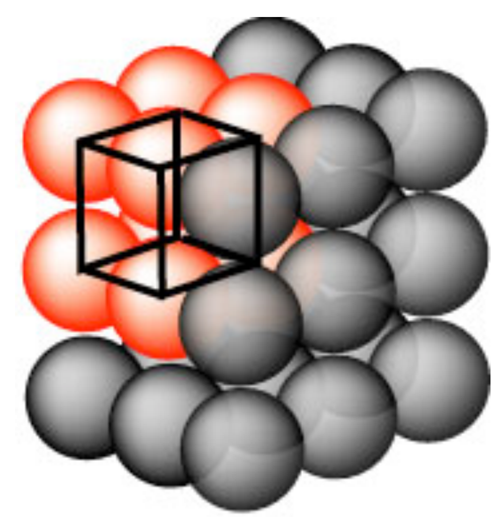
A crystal of atoms breaks translational symmetry of space.



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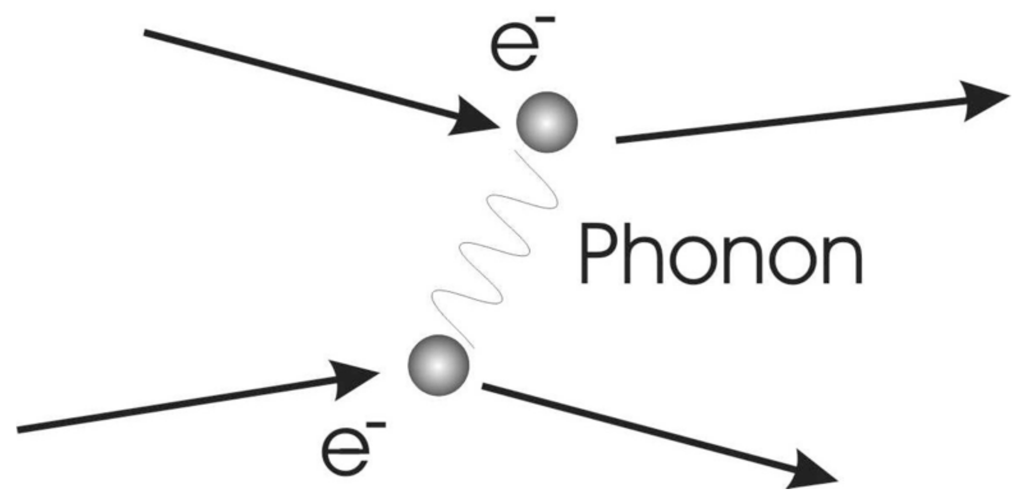
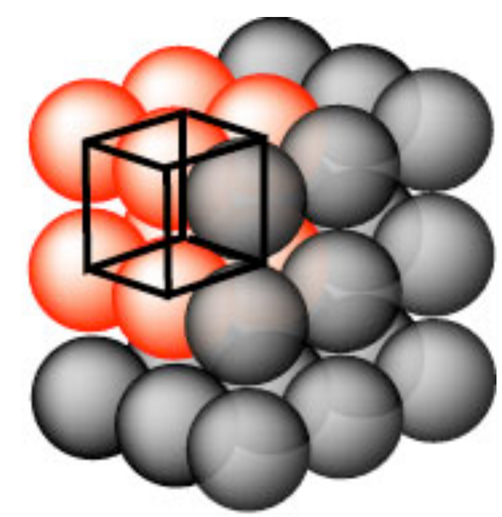
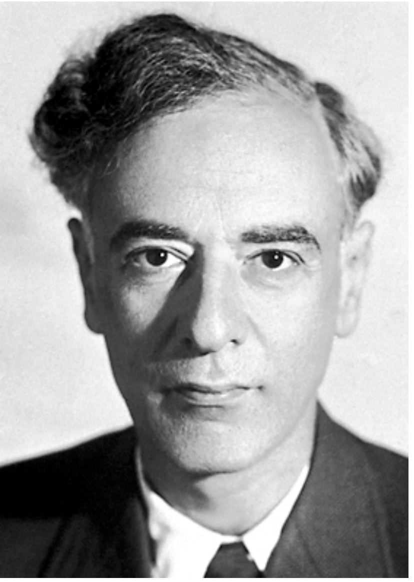


The Goldstone mode of a crystal are lattice vibrations (phonons)!

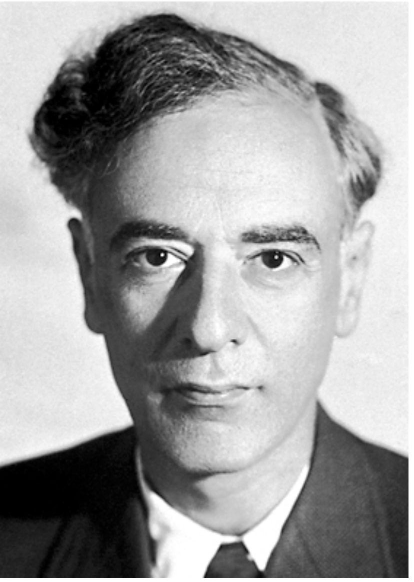


Order parameter



In 1937, L. Landau defined the concept of the order parameter,
which is the expectation value of an observable.
It is non-zero only in the ordered phase.

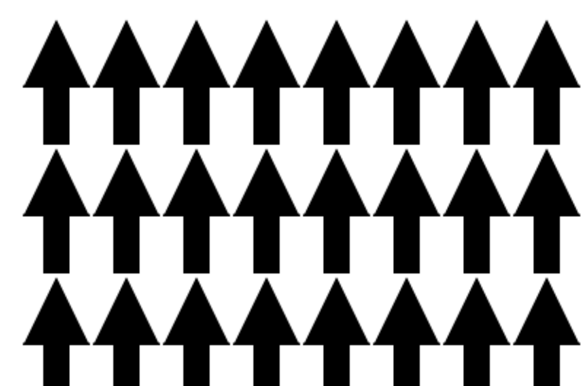
Order parameter



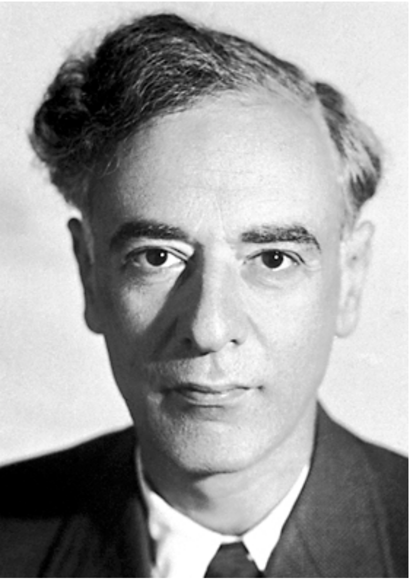
In 1937, L. Landau defined the concept of the order parameter, which is the expectation value of an observable. It is non-zero only in the ordered phase.

In the case of a magnet, the order parameter is a vector set by the local statistical average of the spin,

$$\vec{\phi}(\mathbf{x}) = \langle \mathbf{S}(\mathbf{x}) \rangle$$



Order parameter



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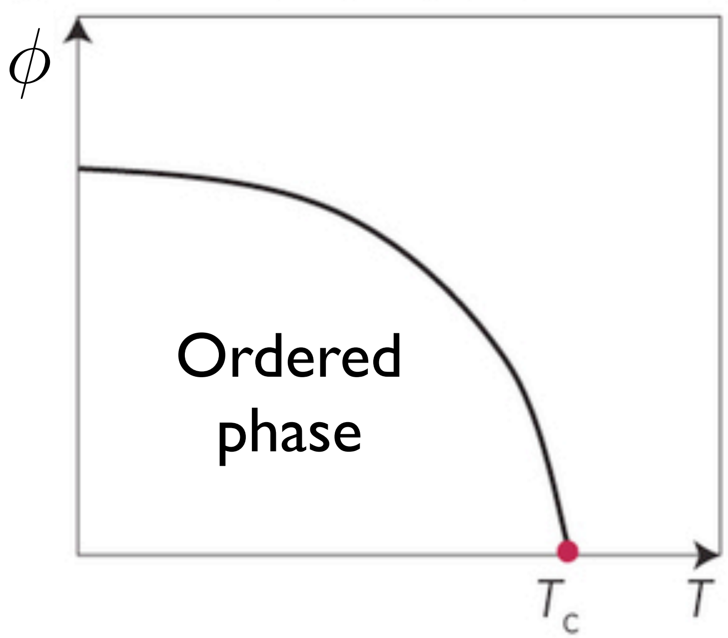
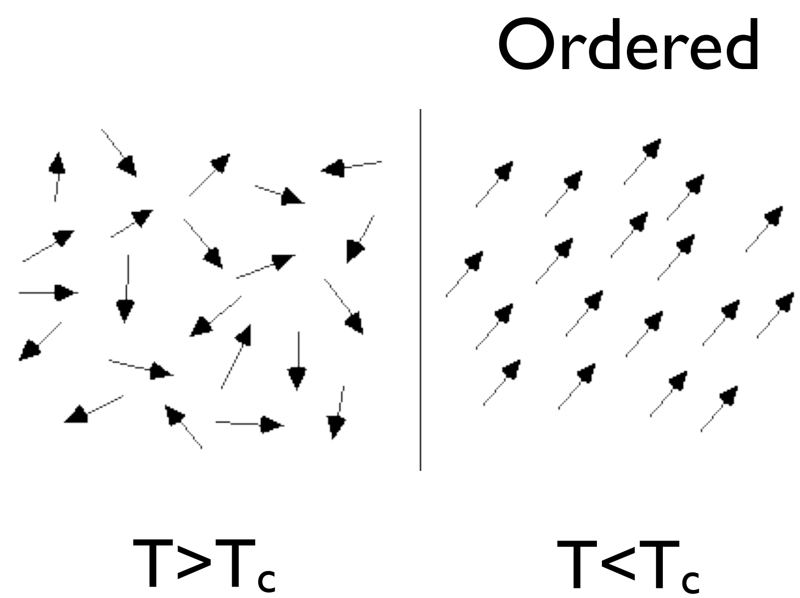
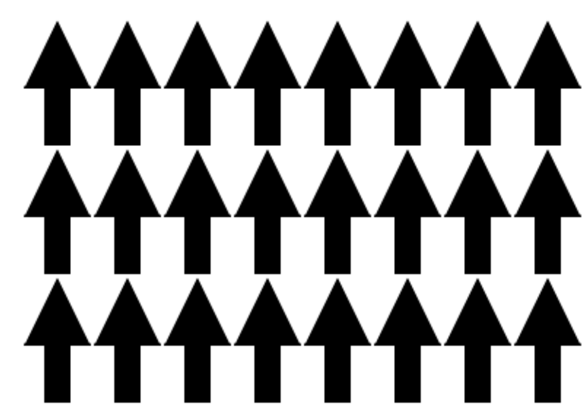
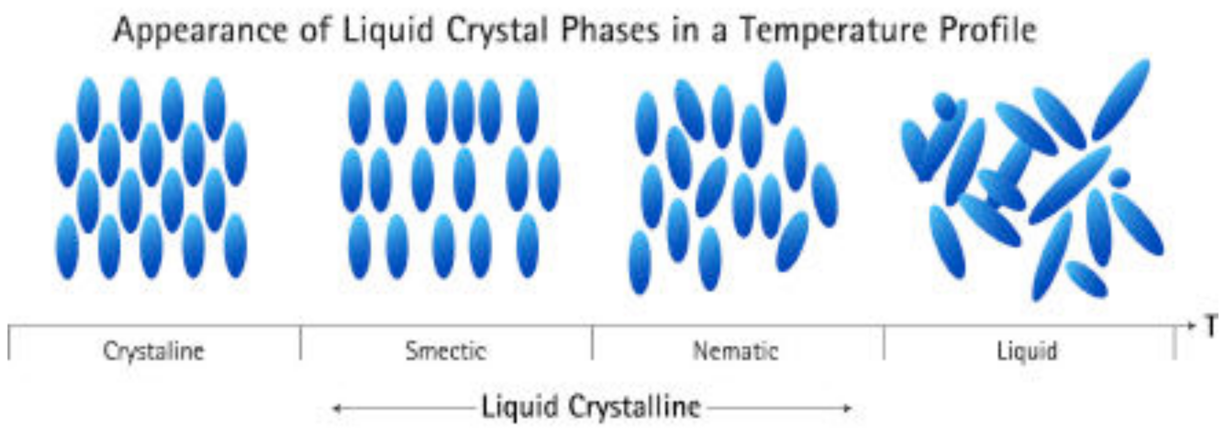


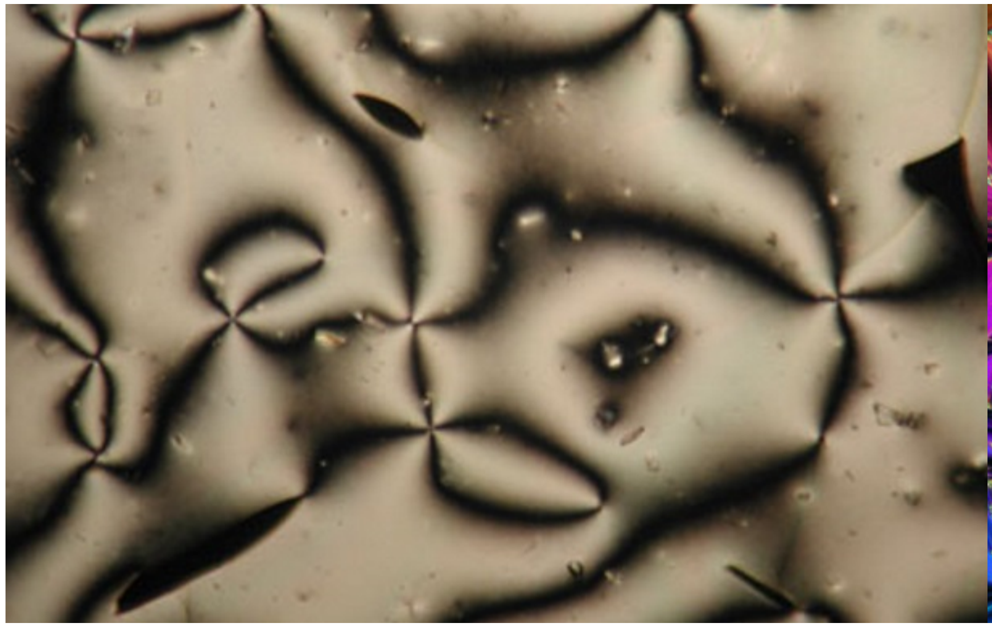


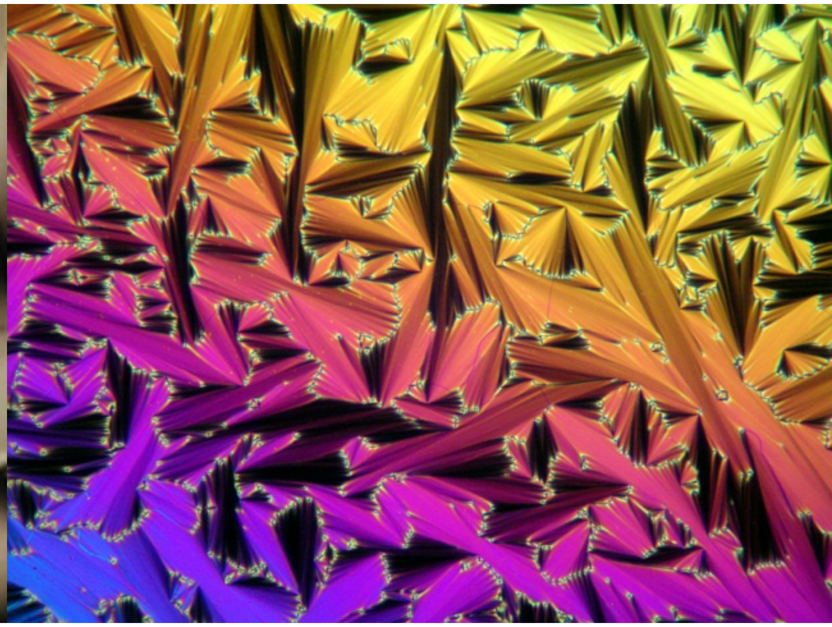
The order parameter is zero at the phase transition



Order parameter

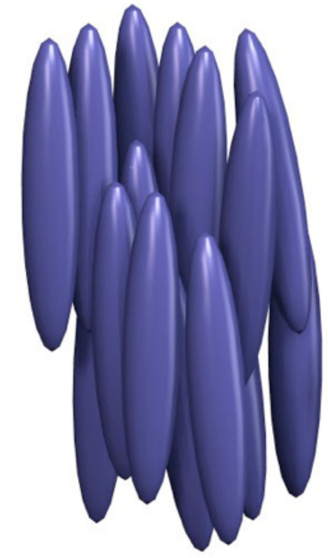




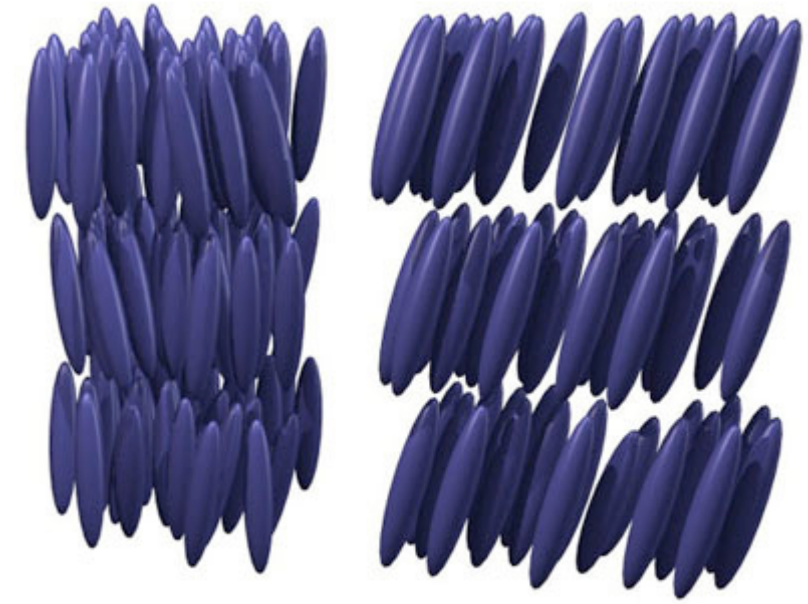


Liquid crystals are soft systems that have some translational symmetry, but are neither solids nor liquids.

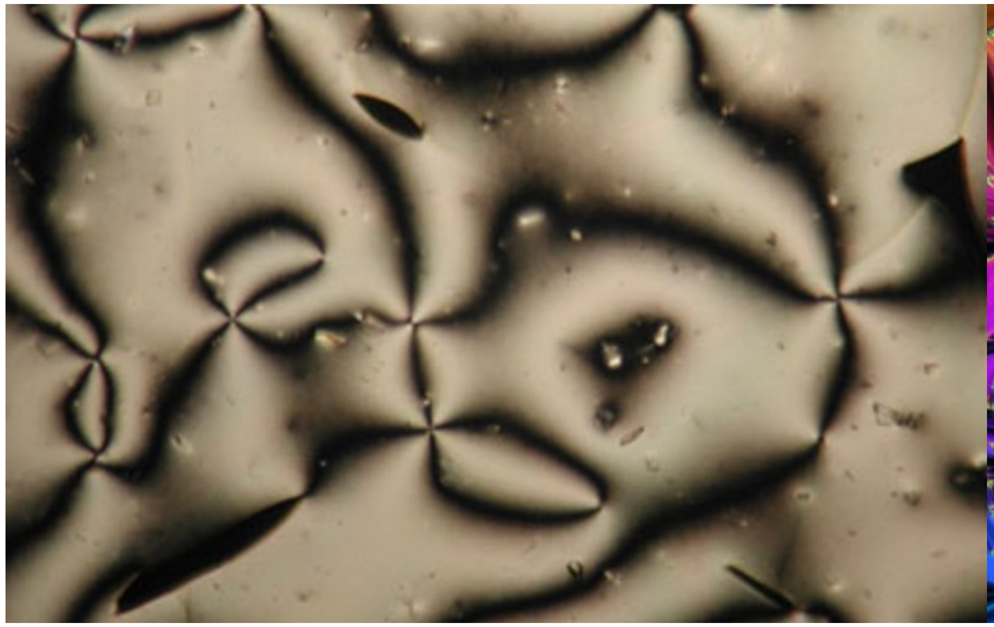
Order parameter

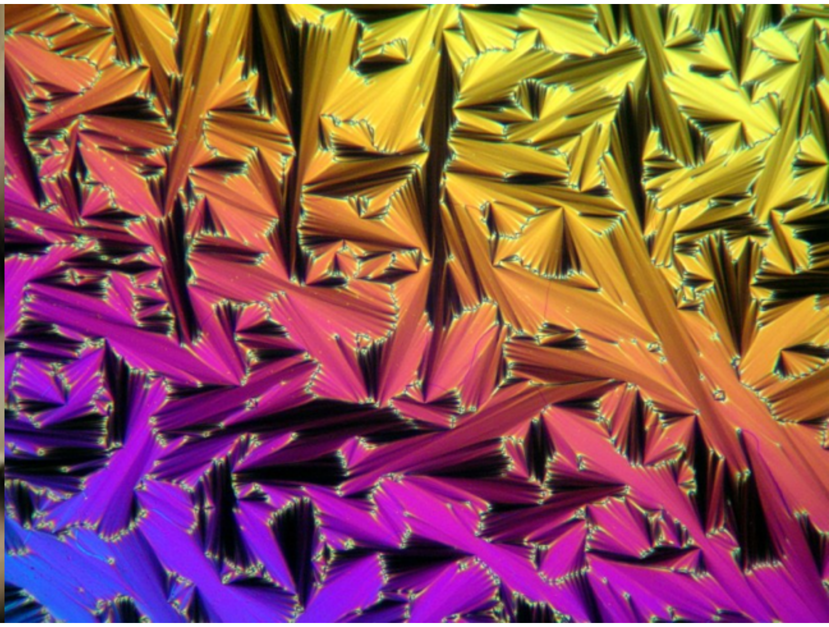


Nematic phase



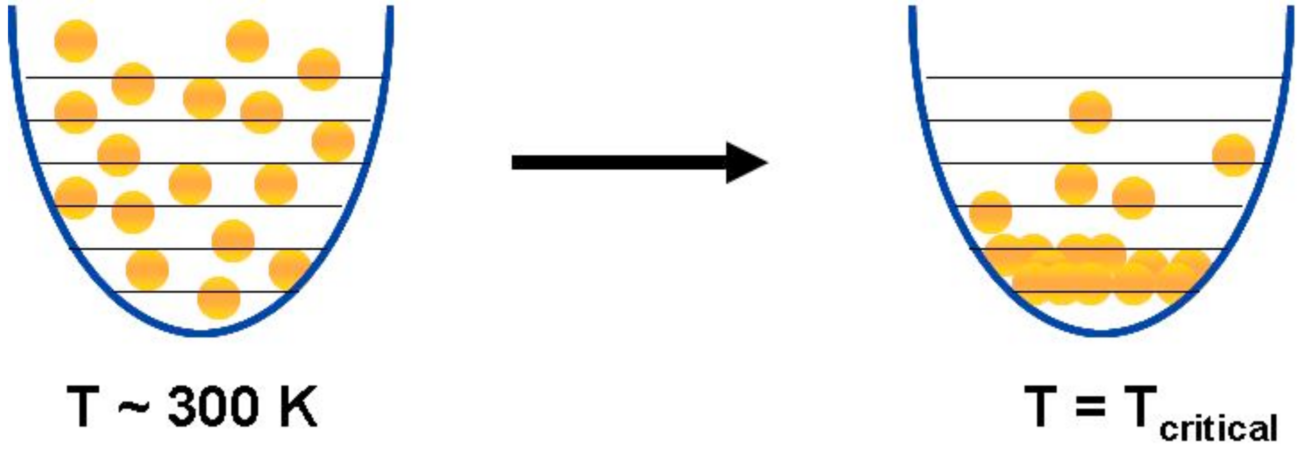
Smetic phase

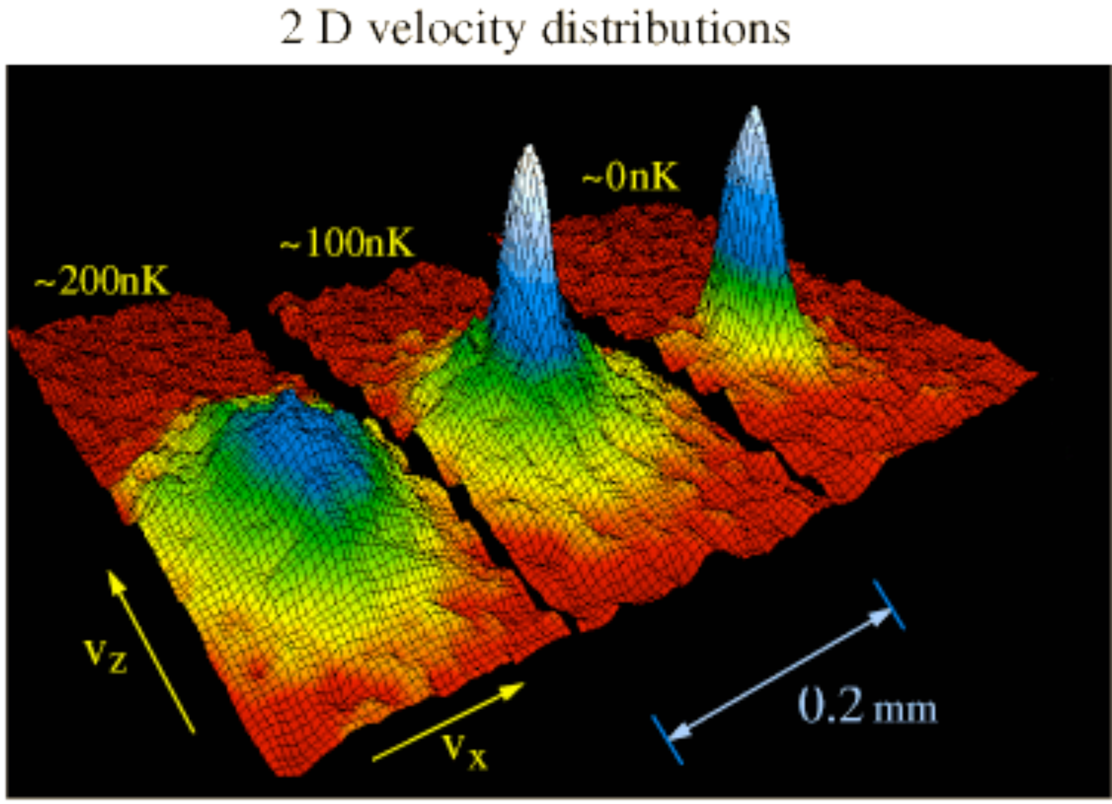




Liquid crystals can have orientational order (nematic) or translational order (smetic). The order parameter is a tensor defined by the average orientation of alignment of the molecules.

Order parameter



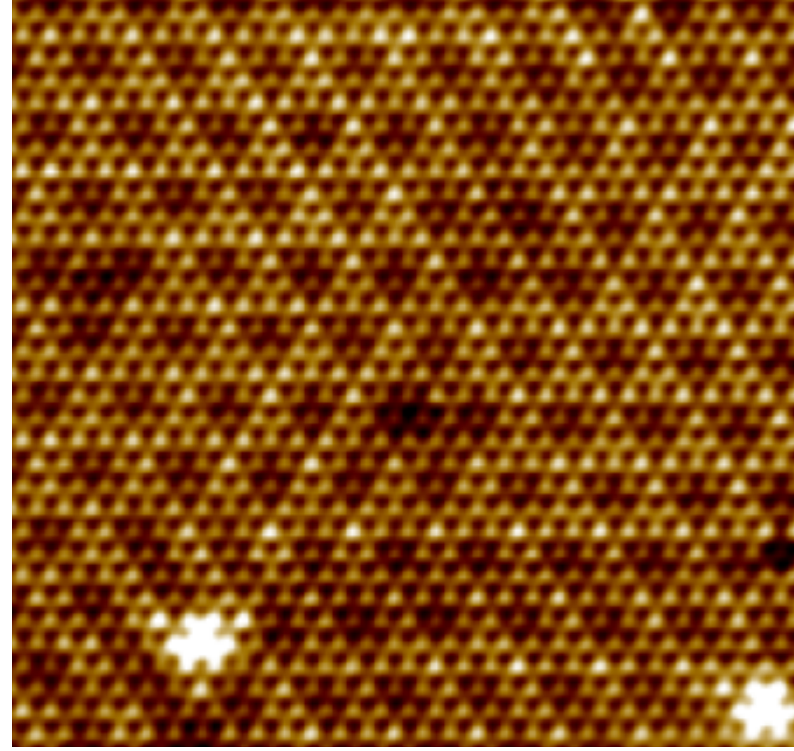


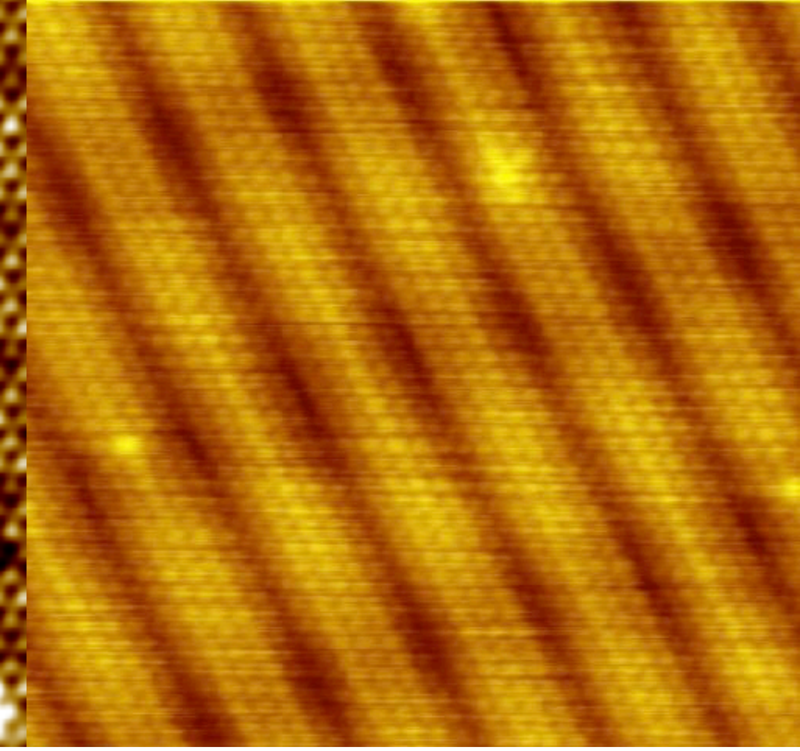
Superfluids have a macroscopic occupation of bosons in the ground energy level. The order parameter for superfluids is a complex number proportional to the macroscopic density of the condensate,

$$\psi(\mathbf{x}) \in \mathbb{C}.$$

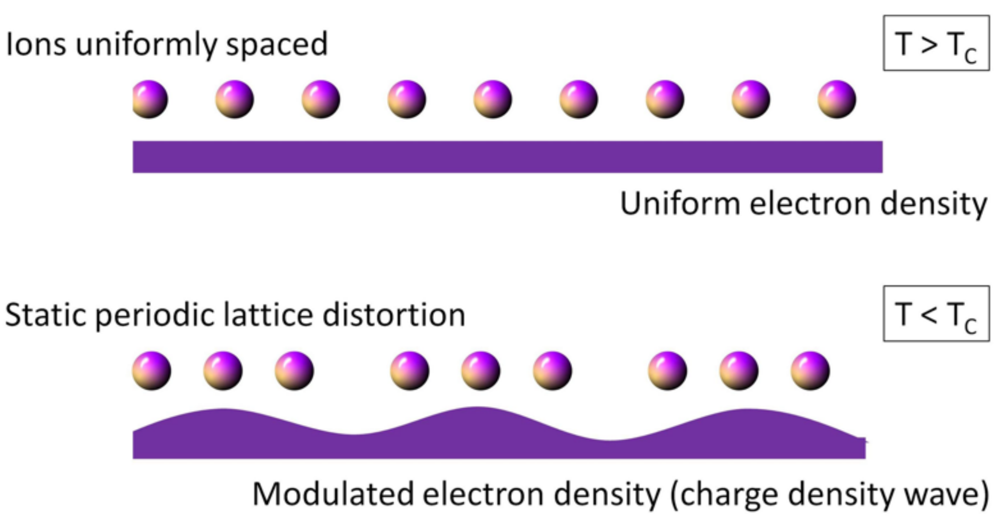
Above the critical temperature, the density of particles in the ground level is zero in the thermodynamic limit.

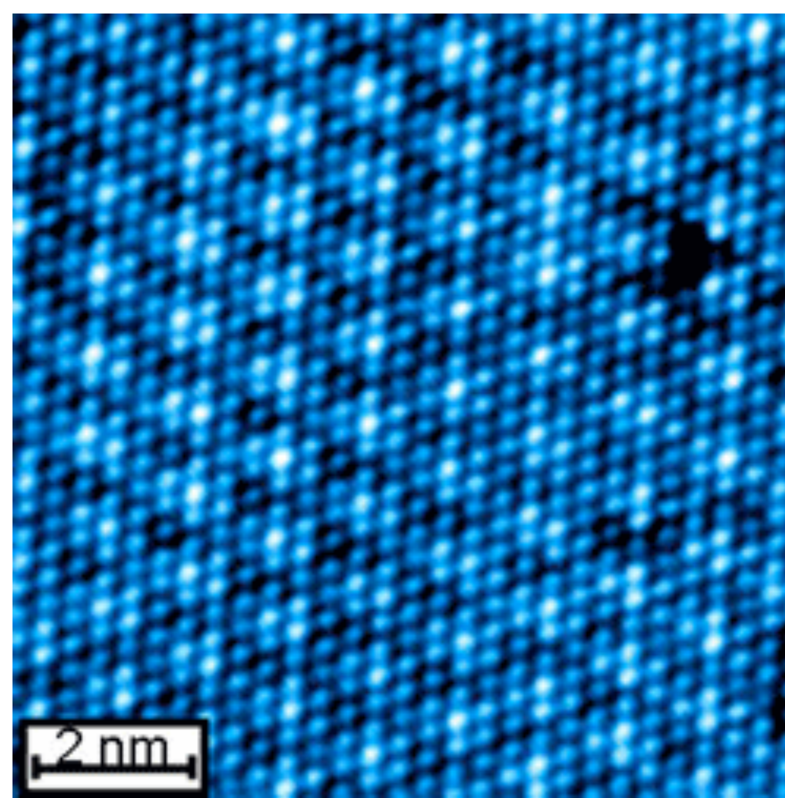
Order parameter

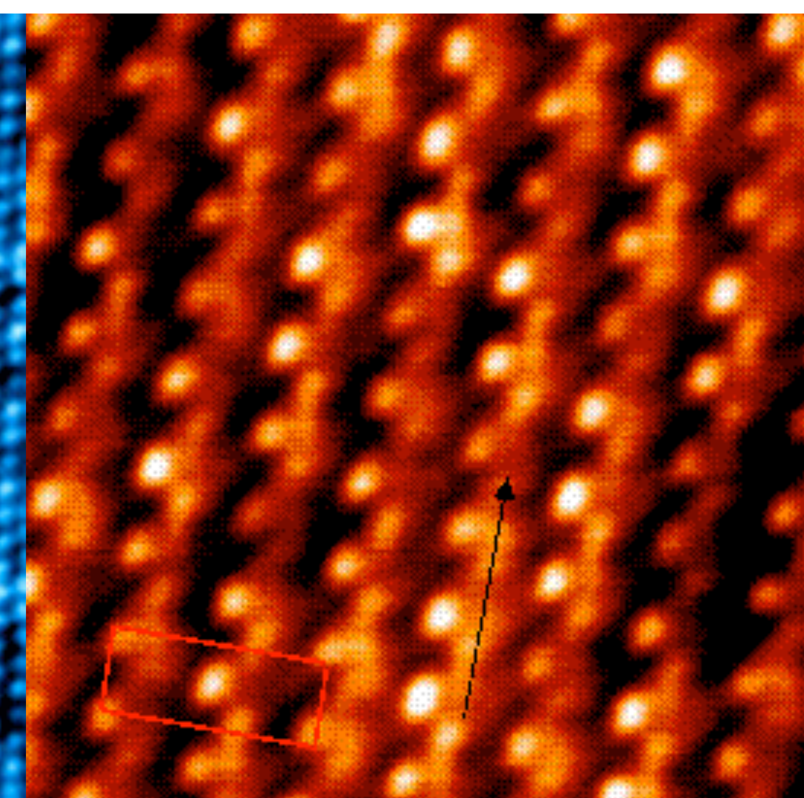




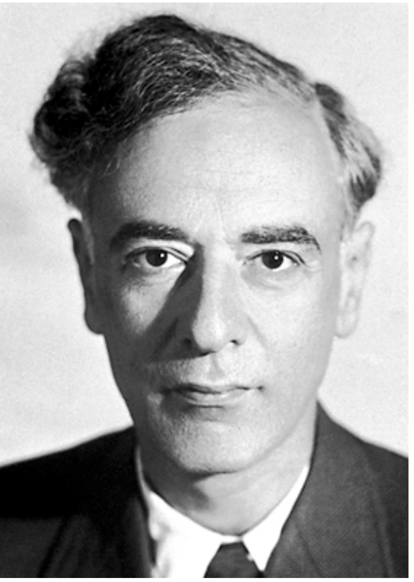
Order parameters with a periodic spacial profile are called density waves. When the period of modulation is infinite, the ordered phase is uniform.







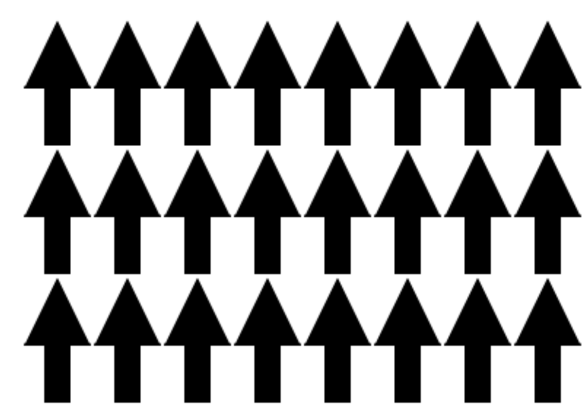
Phase transition



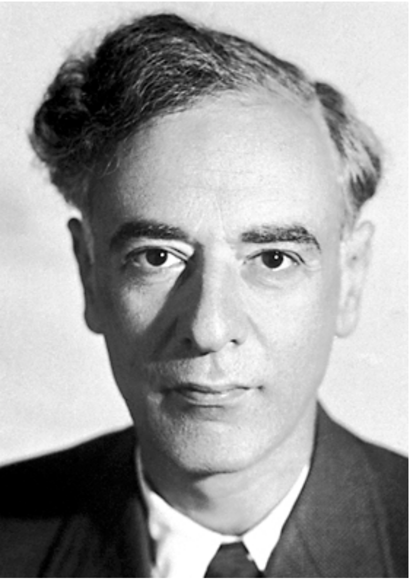
Landau proposed to expand the energy in powers of the order parameter near the phase transition.

$$E(\phi) = -h\phi + r_0\phi^2 + s\phi^3 + u\phi^4 + O(\phi^5)$$

One should keep all possible terms that respect the symmetry of the order parameter



Phase transition



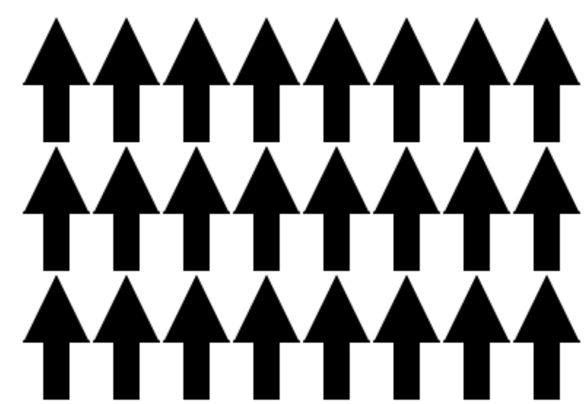
Landau proposed to expand the energy in powers of the order parameter near the phase transition.

$$E(\phi) = -h\phi + r_0\phi^2 + s\phi^3 + u\phi^4 + O(\phi^5)$$

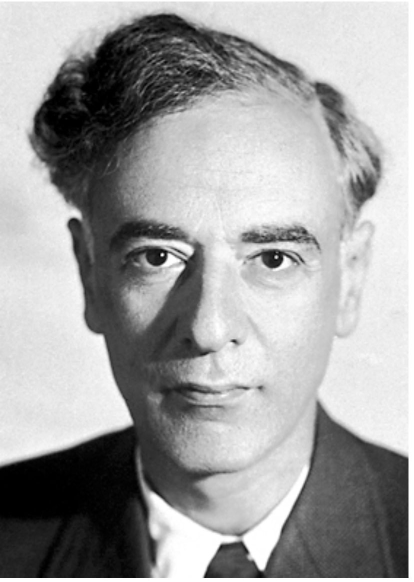
$$\phi = -\frac{\partial E}{\partial h}$$

external field

The external field h is defined as whatever couples with the order parameter in linear response (external magnetic field for spins, etc...)



Phase transition



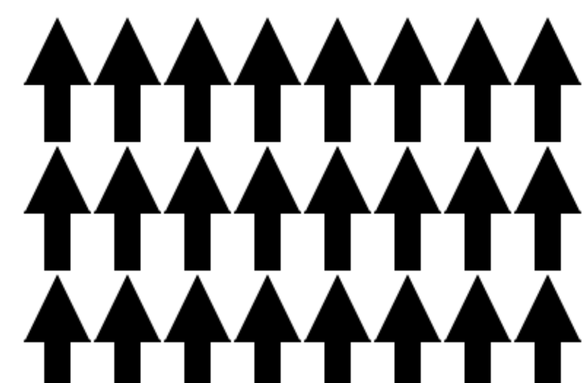
Landau proposed to expand the energy in powers of the order parameter near the phase transition.

$$E(\phi) = -\overset{0}{h}\phi + r_0\phi^2 + \overset{0}{s}\phi^3 + u\phi^4 + O(\phi^5)$$

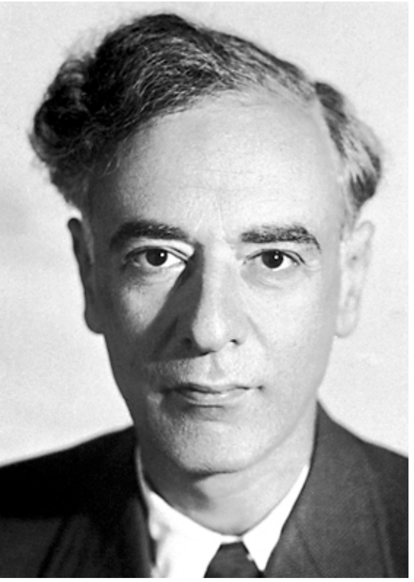
At zero external field, the energy is preserved when

$$\phi \to -\phi$$

since it does not cost any energy to rotate all the spins at the same time, so the coefficients of all odd terms are zero by symmetry!

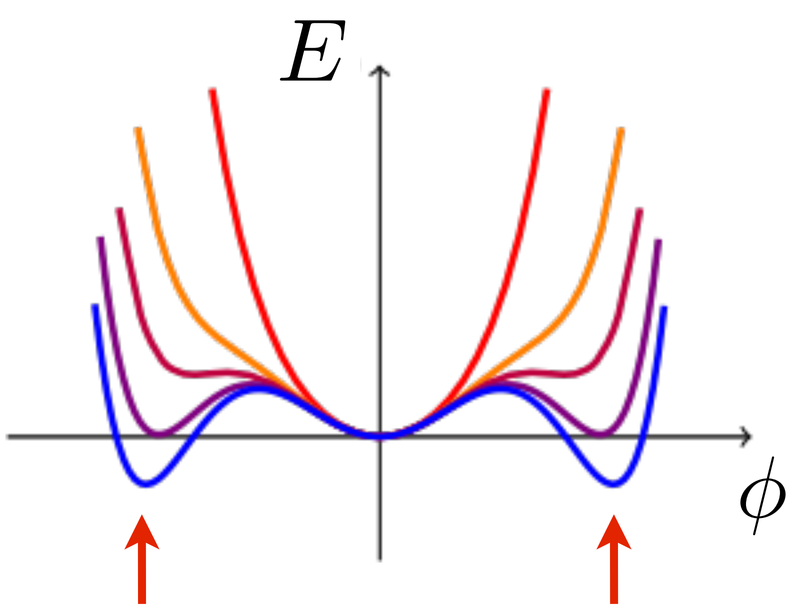


Order parameter



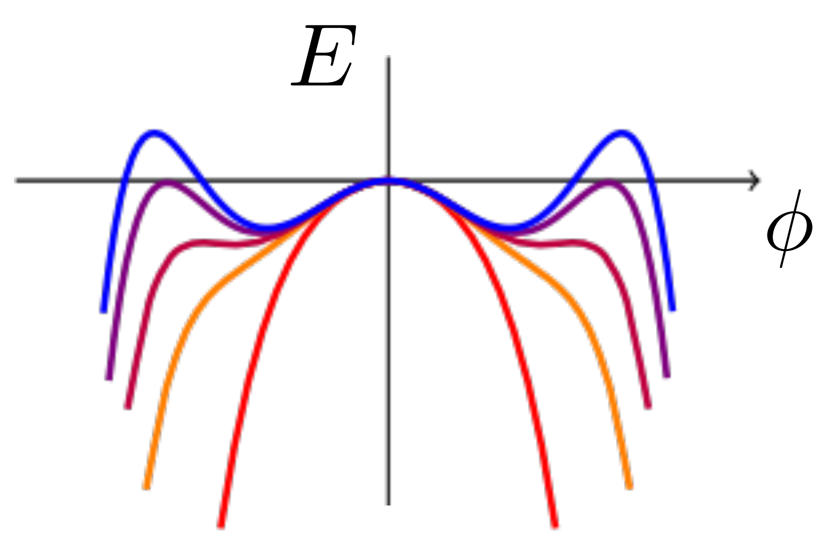
$$E(\phi) = r_0\phi^2 + u\phi^4 + O(\phi^6)$$

$u > 0$ for the broken symmetry state to be a stable minimum!



$u > 0$

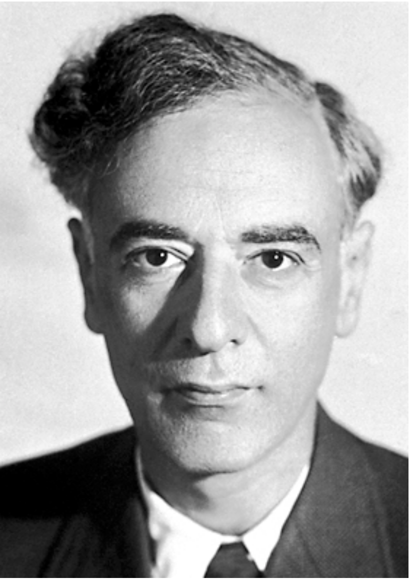
stable ground state



$u < 0$

unstable

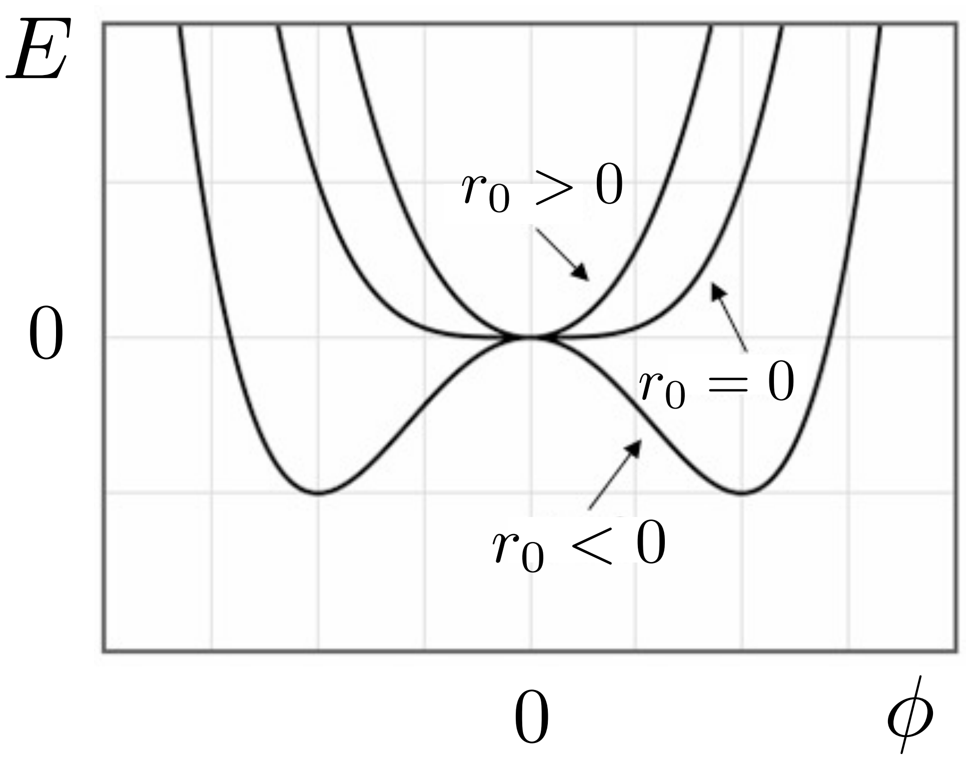
Order parameter



$$E(\phi) = r_0\phi^2 + u\phi^4 + O(\phi^6)$$

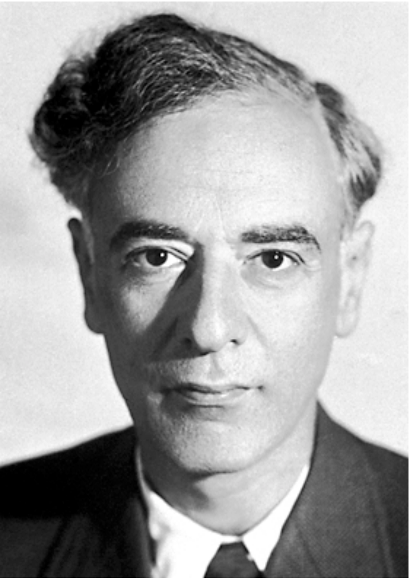
$$\frac{\partial E}{\partial \phi} = \phi(2r_0 + 4u\phi^2) = 0 \longrightarrow \phi = 0 \quad \text{or} \quad \phi = \pm\sqrt{-\frac{r_0}{2u}}$$

Minimum of energy



Broken symmetry state!

Order parameter



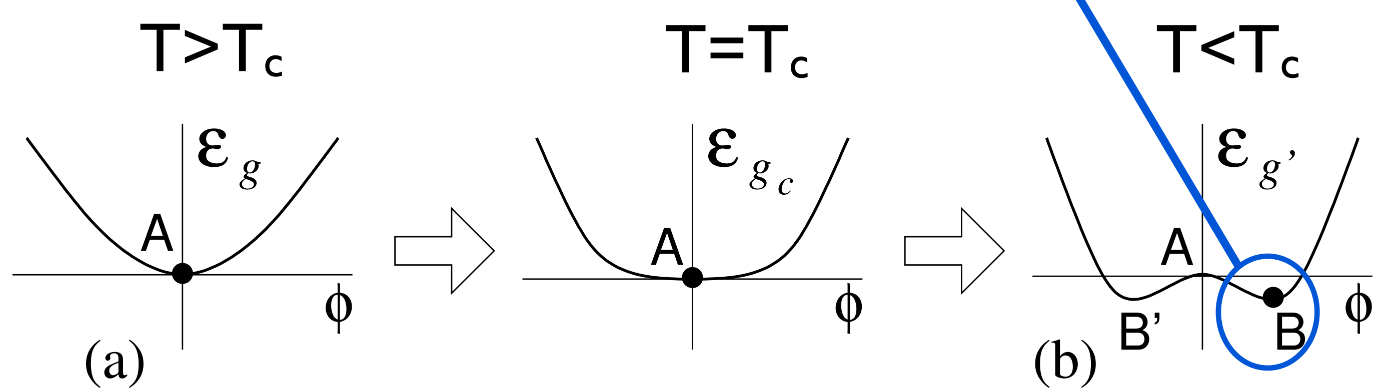
$$E(\phi) = r_0\phi^2 + u\phi^4 + O(\phi^6)$$

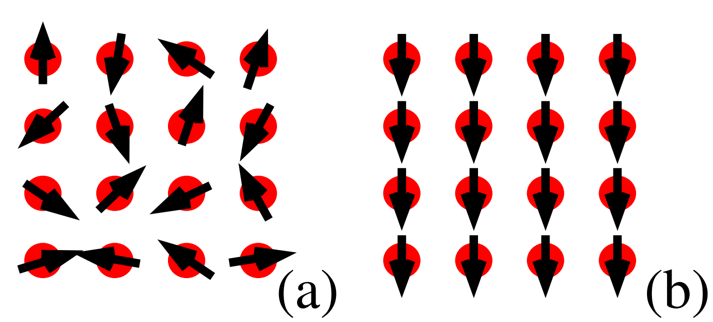
$$r_0 \propto (T - T_c)$$

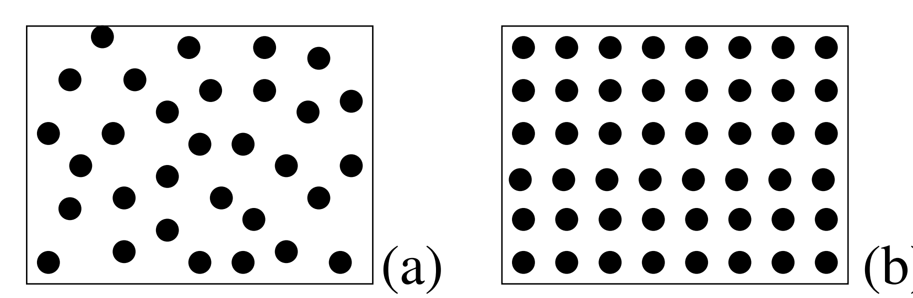
$$> 0$$

$$\phi = \sqrt{-\frac{r_0}{2u}}$$

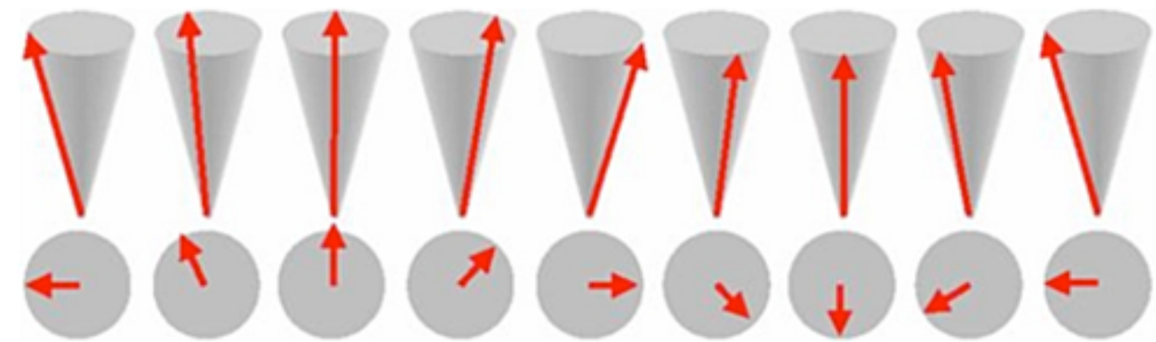
r_0 changes sign at the phase transition!

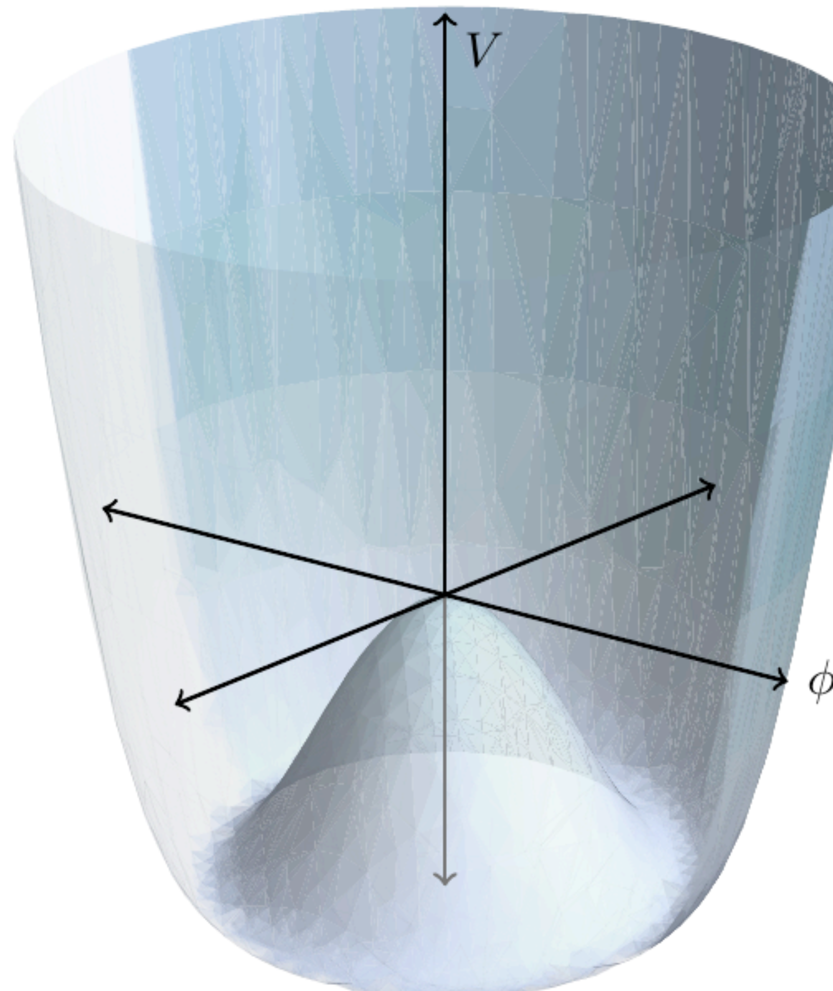


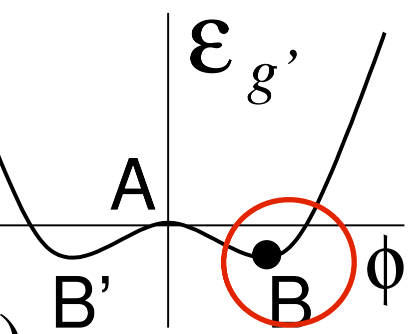




Goldstone theorem



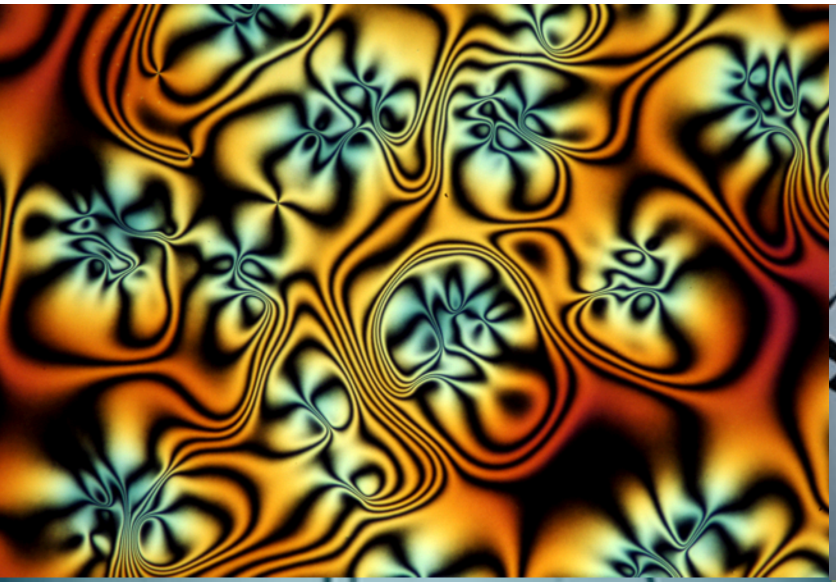




(b)

The ordered state also breaks a continuous rotational symmetry and picks one arbitrary direction in space. The low energy excitations along the minimum of the Mexican hat potential are Nambu-Goldstone modes (spin waves).

This concept can be applied to a variety of phase transitions in completely unrelated systems





In superconductors, the order parameter is the local statistical average of Cooper pairs,

$$\phi(\mathbf{x}) = \langle \psi^\dagger(\mathbf{x}) \psi^\dagger(\mathbf{x}) \rangle$$

In superconductors, the order parameter is the local statistical average of Cooper pairs,

$$\phi(\mathbf{x}) = \langle \psi^\dagger(\mathbf{x}) \psi^\dagger(\mathbf{x}) \rangle$$

Unlike the conventional density of electrons

$$\rho = \langle \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \rangle$$

Superconductivity breaks the gauge symmetry of the phase in the single particle wave function

$$\psi(\mathbf{x}) \to \psi(\mathbf{x}) \mathrm{e}^{i\varphi} \quad \longrightarrow \quad \phi(\mathbf{x}) \to \phi(\mathbf{x}) \mathrm{e}^{-i2\varphi}$$

Goldstone theorem

The spontaneous breaking of any continuous symmetry is associated with a massless and spinless particle known as the Nambu-Goldstone boson.

$\phi(\mathbf{x}) \to \phi(\mathbf{x})\mathrm{e}^{-i2\varphi}$ $\longrightarrow$ What is the Goldstone mode?

The gauge symmetry in the phase of the wavefunction is spontaneously broken in the superconducting state

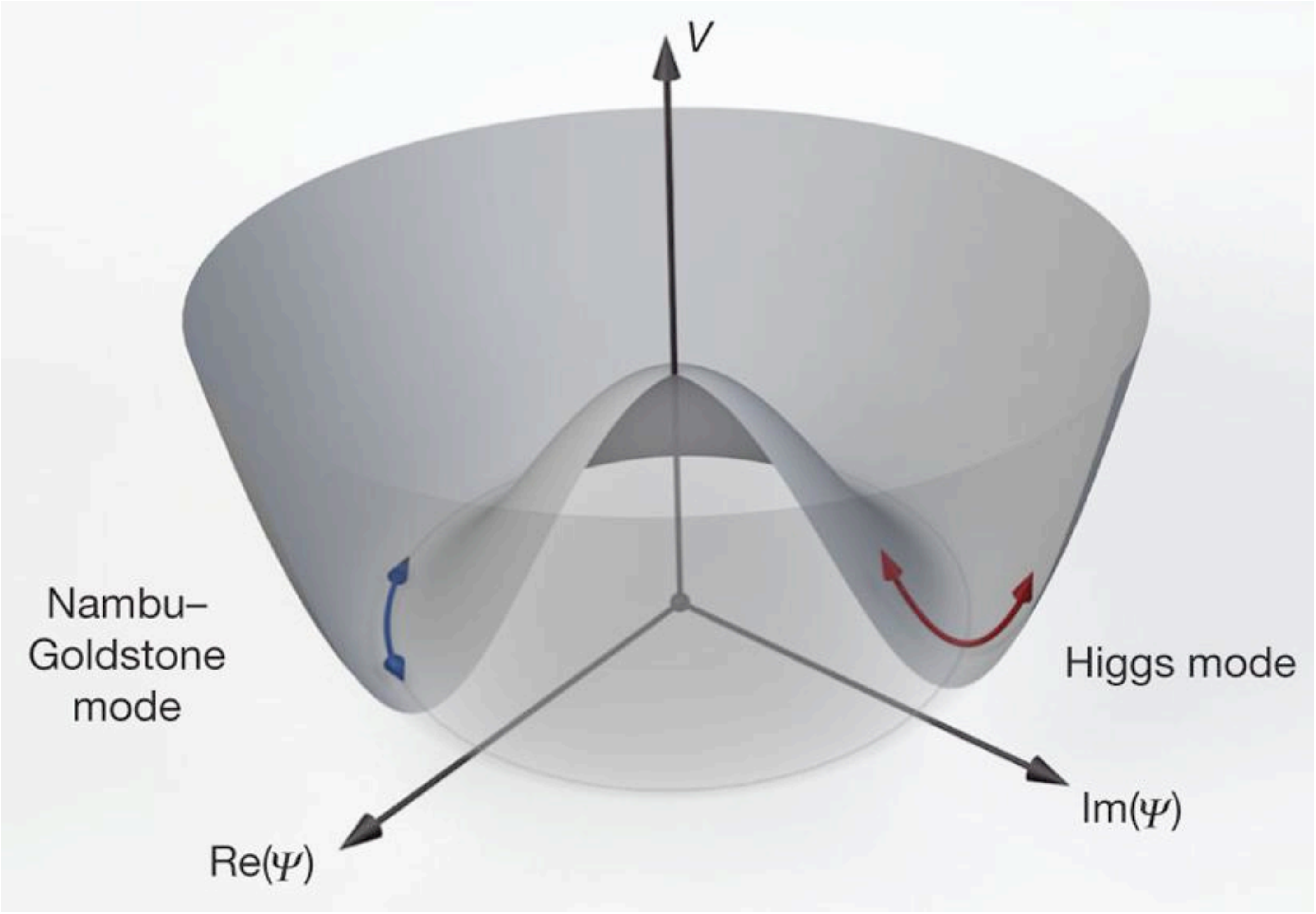
Goldstone theorem

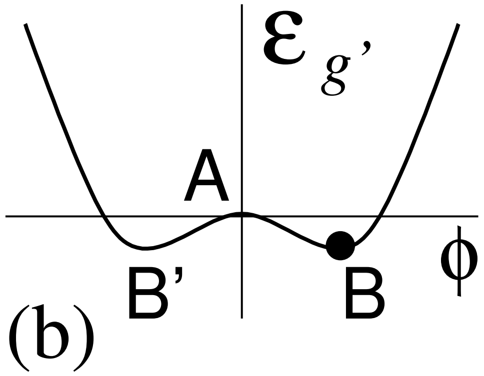
The spontaneous breaking of any continuous symmetry is associated with a massless and spinless particle known as the Nambu-Goldstone boson.

$\phi(\mathbf{x}) \to \phi(\mathbf{x})\mathrm{e}^{-i2\varphi}$ ⟶ It is the phase 2φ itself! (gauge field)

The gauge symmetry in the phase of the wavefunction is spontaneously broken in the superconducting state

Fluctuations of the superconductor order parameter





The Goldstone mode is a phase fluctuation of the order parameter.
The Higgs mode is a density fluctuation in the ordered state.

The Anderson-Higgs mechanism

In 1962, Anderson showed that the coupling of the density fluctuations of a superconductor (Higgs mode) with electromagnetic fields makes photons massive.

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = 0$$

Maxwell's equations
(zero B field)

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \underbrace{e^2\phi_0}_{m}\mathbf{A} = 0$$

In a superconductor

Photons decay inside a superconductor!

The Anderson-Higgs mechanism

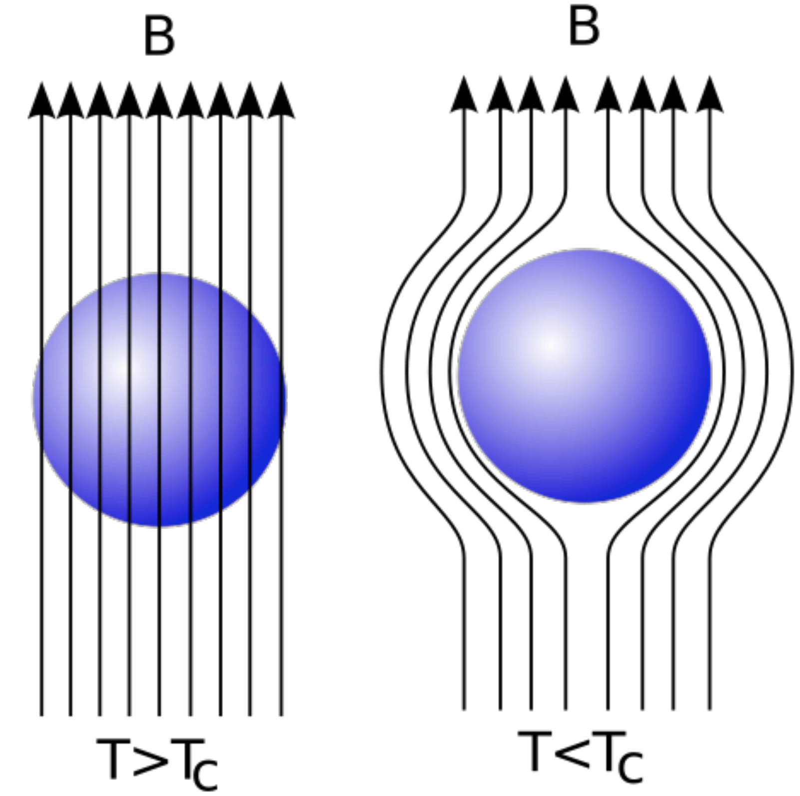
$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} - \frac{4\pi}{c}\mathbf{J} = 0$$

Maxwell's equations
(uniform B field)

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \underbrace{e^2\phi_0}_{m}\mathbf{A} = 0$$

In a superconductor

The Anderson-Higgs mechanism



$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} - \frac{4\pi}{c}\mathbf{J} = 0$$

Maxwell's equations
(uniform B field)

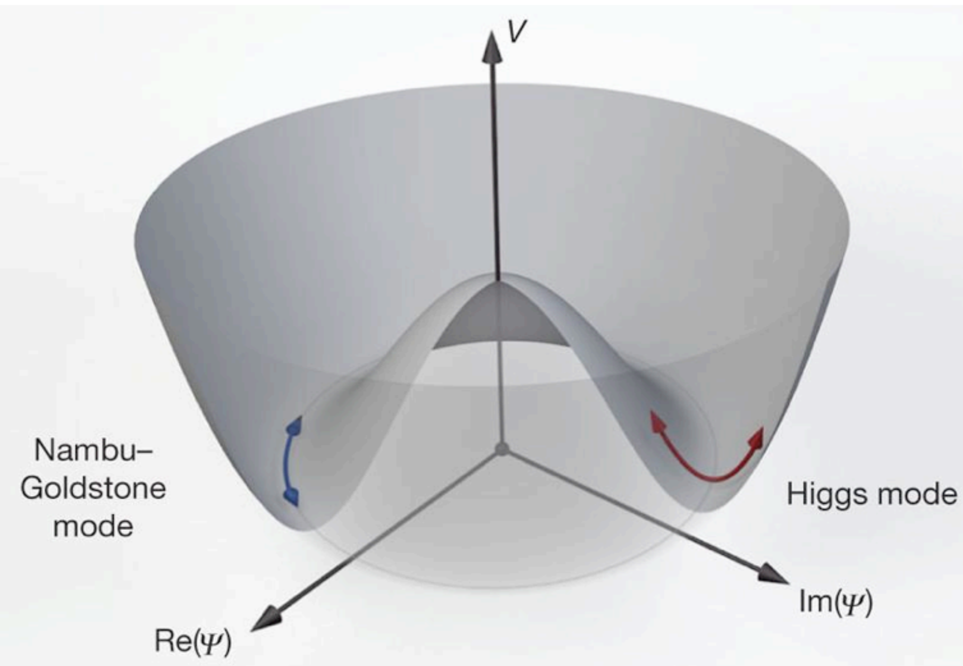
$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \underbrace{e^2\phi_0}_{m}\mathbf{A} = 0$$

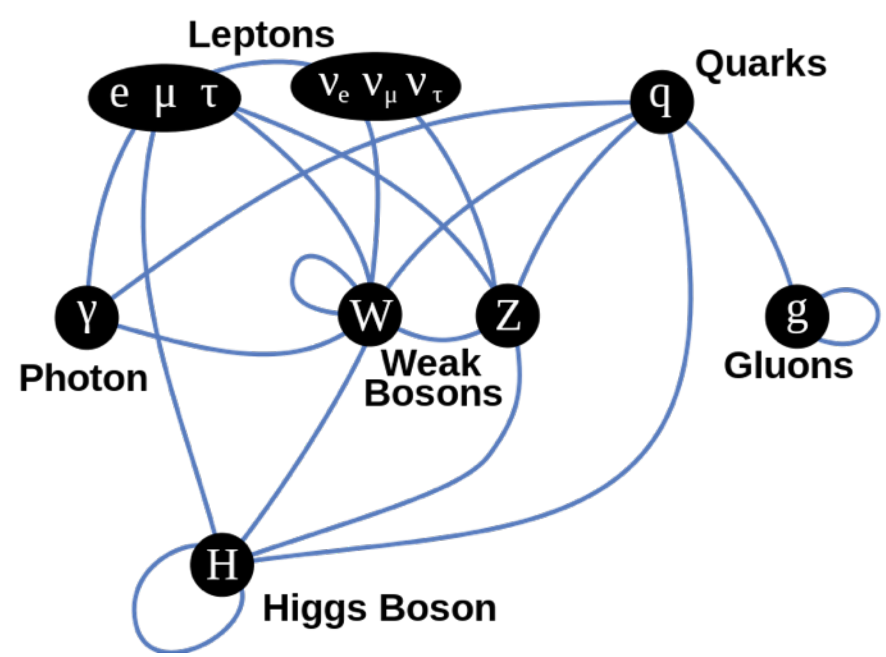
In a superconductor

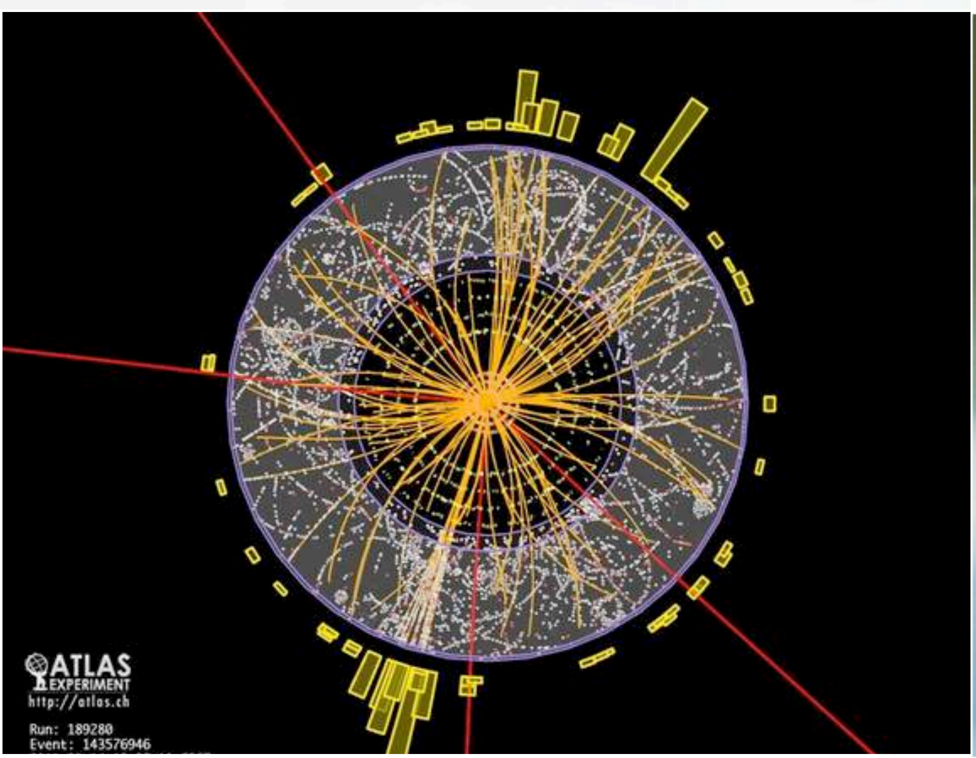
$\boxed{\mathbf{J} \propto \mathbf{A}}$ Diamagnetic current!

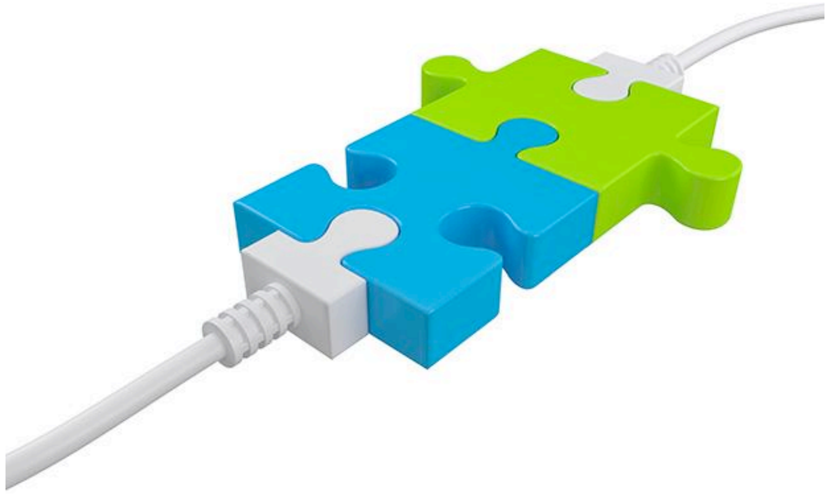
Meissner effect

This phenomenon is similar to the Higgs boson mechanism that creates mass for other particles in the standard model!





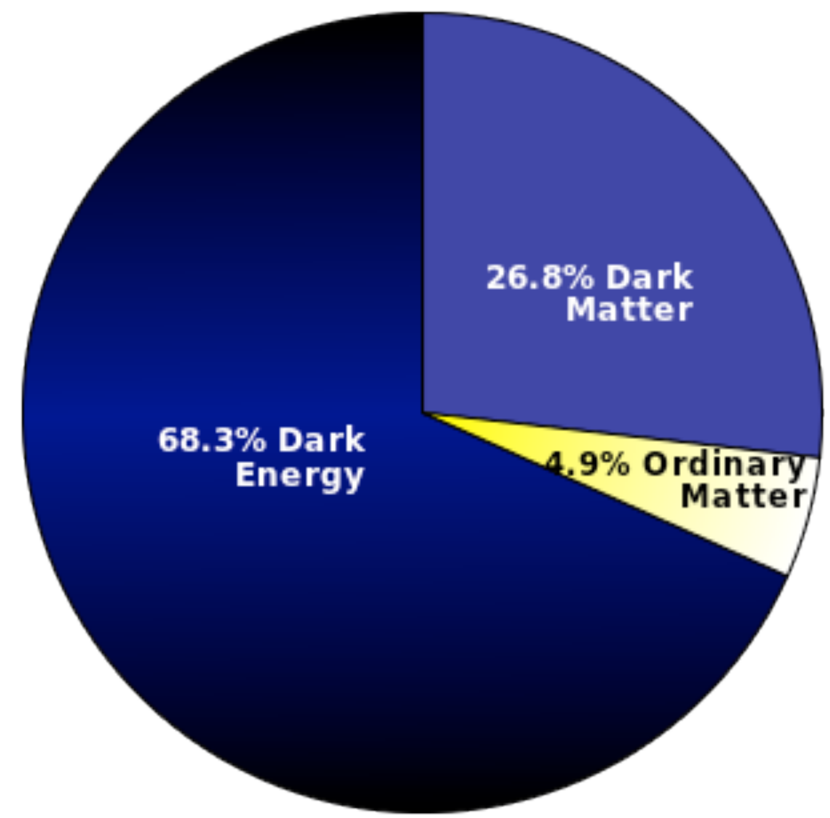




Lesson: Sometimes a physical concept can have applications in different fields! Never stop trying to make connections between fields, even if they seem totally unrelated.

The observed acceleration in the expansion of the universe has been tentatively explained in terms of the existence of an undetected form of energy which permeates the whole universe.

Dark energy?



There are many open puzzles in the universe!