ECE-656: Fall 2011

Lecture 22:

Ionized Impurity Scattering

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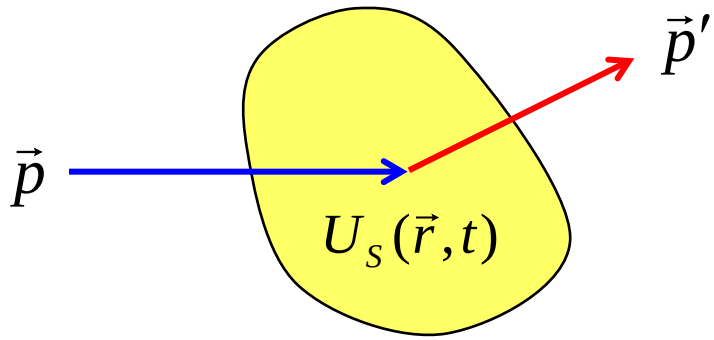
scattering of plane waves

$$\psi_i = \frac{1}{\sqrt{\Omega}} e^{i\vec{p}\bullet\vec{r}/\hbar}$$

incident
plane wave



$$\psi_f = \frac{1}{\sqrt{\Omega}} e^{i\vec{p}'\bullet\vec{r}/\hbar}$$

weak
scattering

$$S(\vec{p},\vec{p}\,') = \frac{2\pi}{\hbar}\left|H_{\vec{p}',\vec{p}}\right|^2 \delta\left(E' - E - \Delta E\right)$$

$$H_{\vec{p}',\vec{p}} = \int_{-\infty}^{+\infty} \psi_f^* U_S(\vec{r}) \psi_i d\vec{r}$$

$$H_{\vec{p}',\vec{p}} = \frac{1}{\Omega}\int_{-\infty}^{+\infty} e^{-i\vec{p}'\bullet r/\hbar} U_S(\vec{r}) e^{i\vec{p}\bullet\vec{r}/\hbar} d\vec{r}$$

infrequent
scattering

examples

"short range potential"

$$U_S(\vec{r}) = C\,\delta(0)$$

$$S(\vec{p},\vec{p}') = K\frac{1}{\Omega}\delta(E'-E)$$

$$\frac{1}{\tau(E)} = \frac{1}{\tau_m(E)} \propto D_f(E)$$

"oscillating, propagating potential"

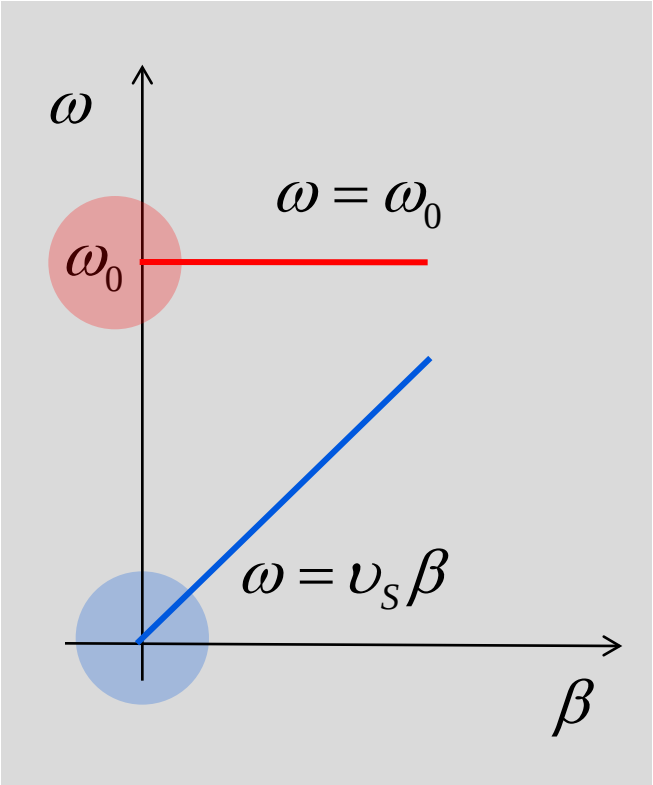
$$U_S(\vec{r},t) = \frac{U_\beta^{a,e}}{\sqrt{\Omega}} e^{\pm i\left(\vec{\beta}\bullet\vec{r}-\omega t\right)}$$

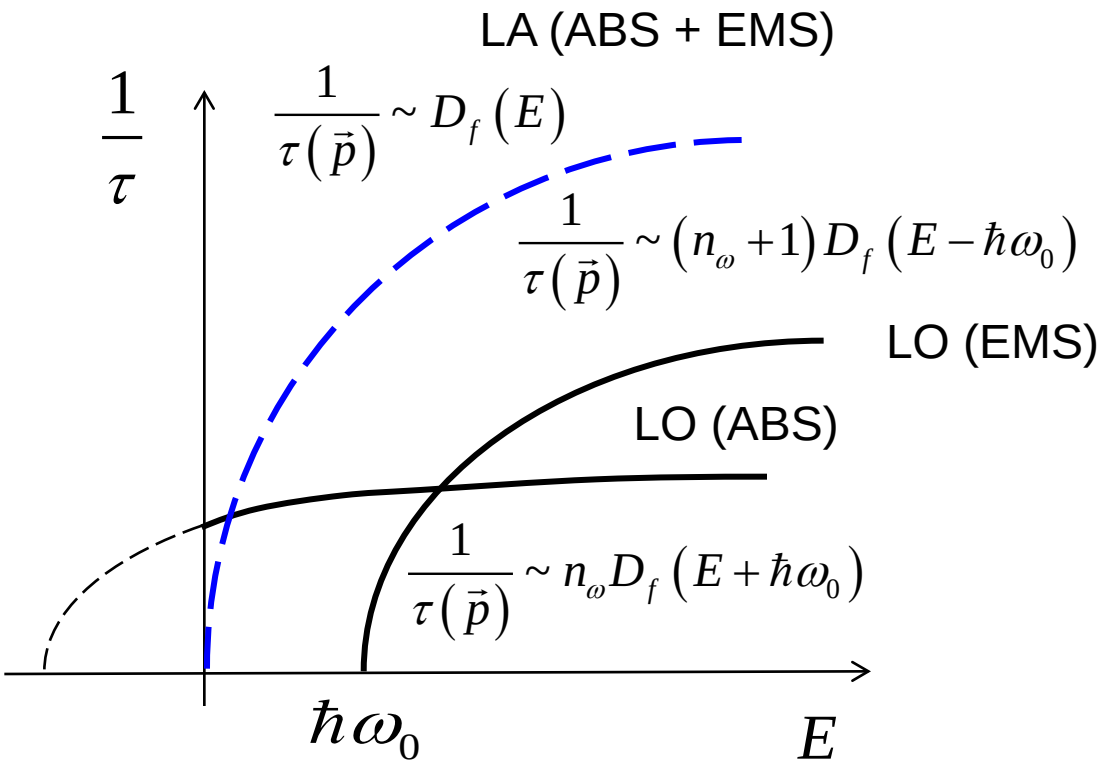
$$S(\vec{p},\vec{p}') = \frac{2\pi}{\hbar}\frac{\left|U_\beta^{a,e}\right|^2}{\Omega}\delta\left(E'-E\mp\hbar\omega\right)\delta_{\vec{p}',\vec{p}\pm\hbar\vec{\beta}}$$

$$\frac{1}{\tau(E)} = \frac{1}{\tau_m(E)} \propto D_f(E\pm\hbar\omega)$$

$$\frac{1}{\tau_E(\vec{p})} = \left(\frac{\hbar\omega}{E}\right)\frac{1}{\tau(\vec{p})}$$

acoustic vs. optical phonon scattering

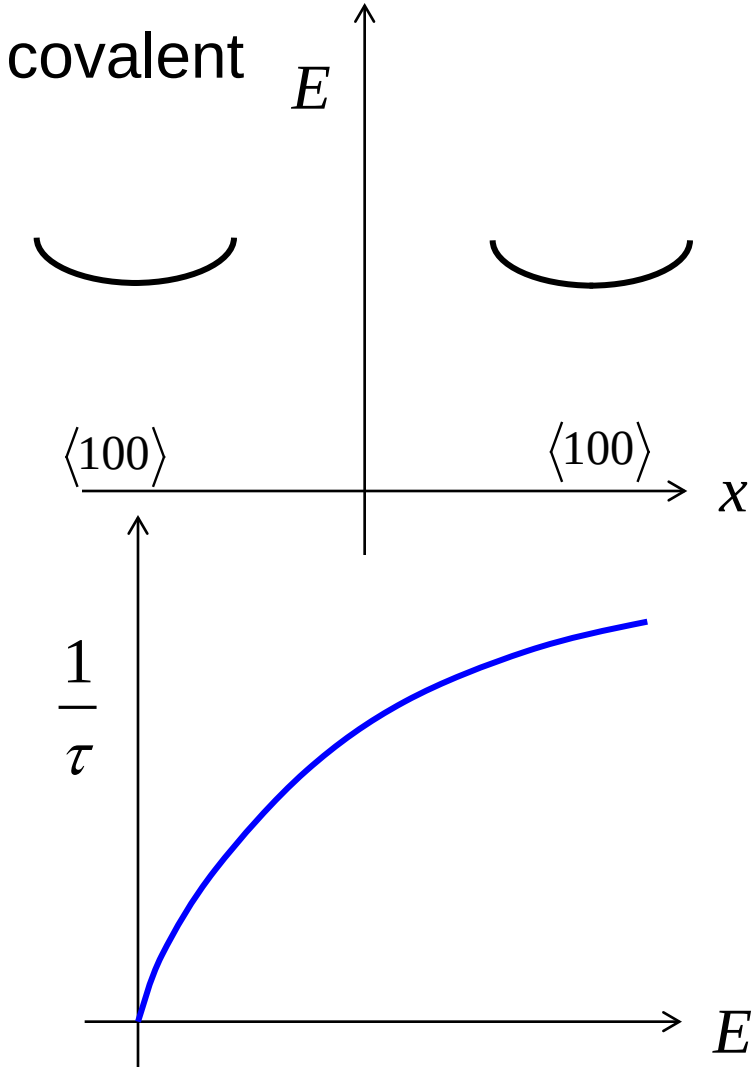


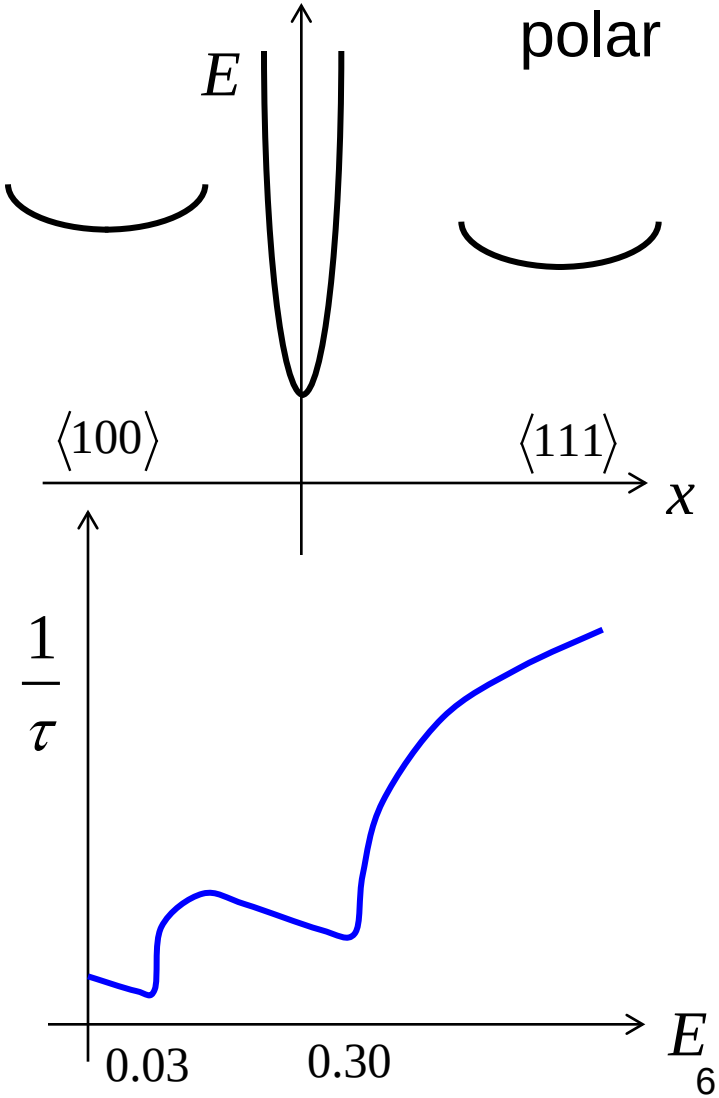


summary

1) Characteristic times are derived from the transition rate, $S(p,p')$
2) $S(p,p')$ is obtained from Fermi's Golden Rule
3) The scattering rate is proportional to the final DOS
4) Static potentials lead to elastic scattering
5) Time varying potentials lead to inelastic scattering
6) General features of scattering in common semiconductors can now be understood (almost)

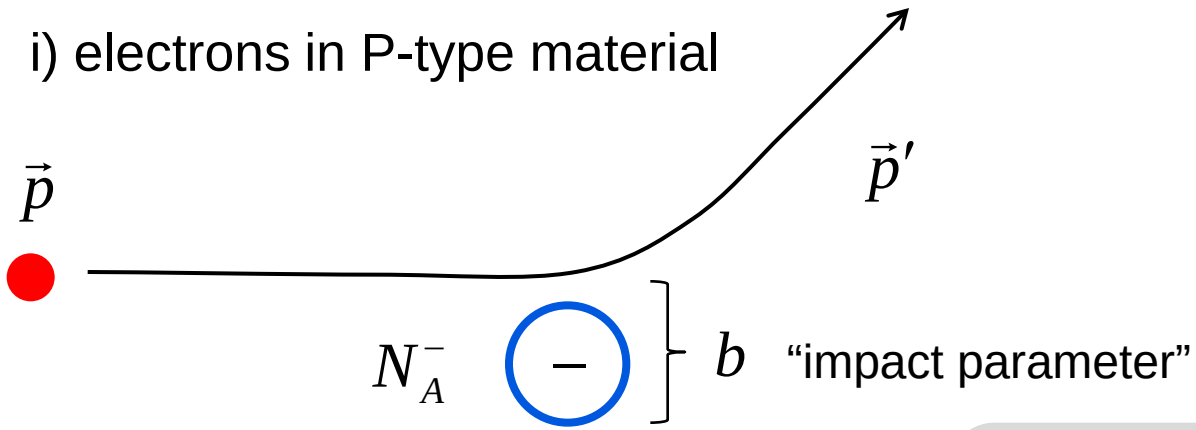
covalent vs. polar semiconductors



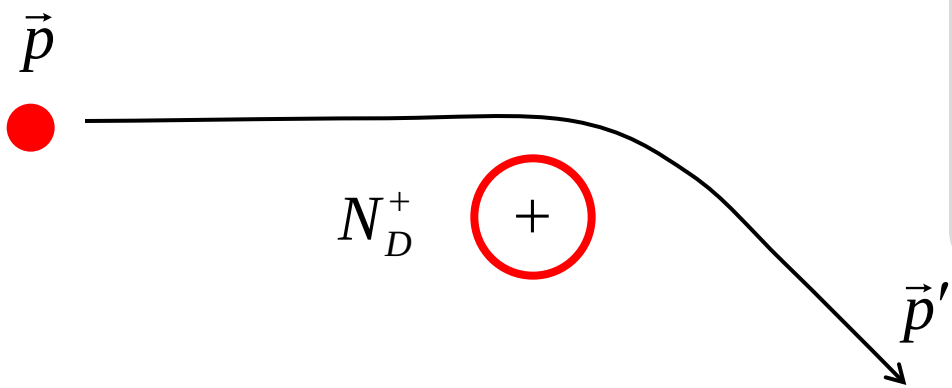


II scattering potential

i) electrons in P-type material



ii) electrons in N-type material



$$U_S(\vec{r}) = \pm \frac{q^2}{4\pi\kappa_S\varepsilon_0 r}$$

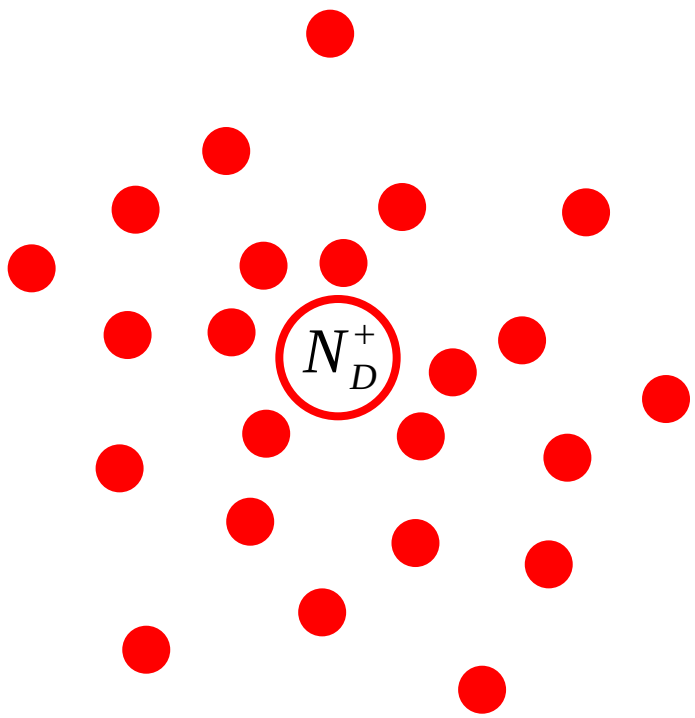
According to FGR, the transition rate is independent of the sign of the scattering potential.

outline

1) Review
2) **Screening**
3) Brooks-Herring approach
4) Conwell-Weisskopf approach
5) Discussion
6) Summary / Questions

(Reference: Chapter 2, Lundstrom, FCT)

screening



Bare Coulomb potential.

$$U_S(\vec{r}) = \frac{q^2}{4\pi\kappa_S\varepsilon_0 r}$$

Screened Coulomb potential: ??

Mobile charges attracted to fixed charges “screen” out the fixed charge.

screening in 3D

$$\nabla^2 V(\vec{r}) = -\frac{\rho}{\kappa_S \varepsilon_0} = -\frac{q\left[N_D^+ - n(\vec{r})\right]}{\kappa_S \varepsilon_0}$$

$$n(\vec{r}) \approx N_D^+ = n_0$$

$$V(\vec{r}) = V_0$$

$$\delta n(\vec{r}) \approx n(\vec{r}) - n_0$$

$$\delta V(\vec{r}) \approx V(\vec{r}) - V_0$$

$$\nabla^2 \delta V(\vec{r}) = -\frac{q}{\kappa_S \varepsilon_0} \frac{\partial n_0(\vec{r})}{\partial V} \delta V(\vec{r})$$

$$\frac{1}{L_D^2} \equiv \frac{q}{\kappa_S \varepsilon_0} \frac{\partial n_0(\vec{r})}{\partial V}$$

$$n_0(\vec{r}) = \frac{1}{\Omega} \sum_{\vec{k}} f_0(k)$$

$$f_0(k) = \frac{1}{1 + e^{\left(E_C(\vec{r}) + E(\vec{k}) - E_F\right)/k_B T}}$$

$$\frac{\partial n_0(\vec{r})}{\partial V} = q \frac{\partial n_0(\vec{r})}{\partial E_F}$$

$$\frac{1}{L_D^2} \equiv \frac{q^2}{\kappa_S \varepsilon_0} \frac{\partial n_0(\vec{r})}{\partial E_F}$$

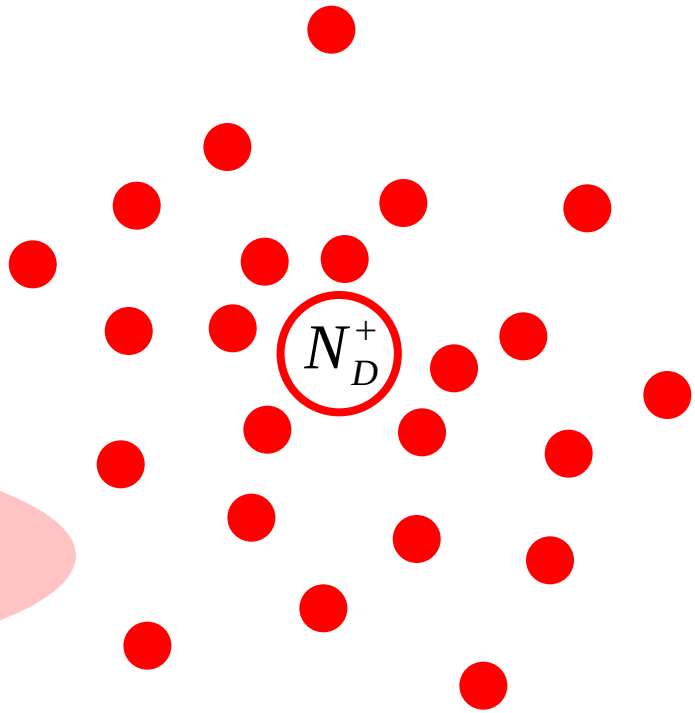
screening in 3D

$$\nabla^2 \delta V(\vec{r}) = \frac{1}{L_D^2}\delta V(\vec{r}) \qquad \frac{1}{L_D^2} \equiv \frac{q^2}{\kappa_S \varepsilon_0}\frac{\partial n_0(\vec{r})}{\partial E_F}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = \frac{1}{L_D^2}\delta V(\vec{r})$$

$$\delta V(r) = C\frac{e^{-r/L_D}}{r}$$

$$U_S(r) = -q\delta V(r) = \frac{q^2}{4\pi\kappa_S \varepsilon_0 r}e^{-r/L_D}$$



Debye length in 3D

$$U_S(r) = \frac{q^2}{4\pi\kappa_S\varepsilon_0 r} e^{-r/L_D}$$

$$\frac{1}{L_D^2} \equiv \frac{q^2}{\kappa_S\varepsilon_0}\frac{\partial n_0(\vec{r})}{\partial E_F} = \frac{q^2}{\kappa_S\varepsilon_0 k_B T}\frac{\partial n_0(\vec{r})}{\partial \eta_F}$$

$$n_0 = N_{3D}\mathcal{F}_{1/2}(\eta_F)$$

$$L_D = \sqrt{\frac{\kappa_S\varepsilon_0 k_B T_L}{q^2 n_0}}$$

Debye length

(non-degenerate)

$$\frac{\partial n_0}{\partial \eta_F} = N_{3D}\mathcal{F}_{-1/2}(\eta_F) = n_0\frac{\mathcal{F}_{-1/2}(\eta_F)}{\mathcal{F}_{1/2}(\eta_F)} = n_0$$

(non-degenerate)

comments on screening

1) Our semi-classical approach assumes that the potential is slowly varying on the scale of the electron's wavelength. For rapidly varying potentials, a more sophisticated approach is needed. (See Ashcroft and Mermin, pp. 340-343 for a discussion of the Lindhard theory.)

2) Our semi-classical approach also assumes that the potential is slowly in time. (See Ashcroft and Mermin, p. 344 for a brief discussion.)

3) For potentials that vary rapidly in space and time, a "dynamic screening" treatment is needed. (See chapter 9 in Ridley, *Quantum Processes in Semiconductors*, 4th Ed. and Chapter 10 in Ridley, *Electrons and Phonons in Semiconductor Multilayers*.)

4) Screening is generally less effective in 2D and in 1D. (See J.H. Davies, *The Physics of Low-Dimensional Structures*, pp. 350-356

outline

1) Review
2) Screening
3) **Brooks-Herring approach**
4) Conwell-Weisskopf approach
5) Discussion
6) Summary / Questions

(Reference: Chapter 2, Lundstrom, FCT)

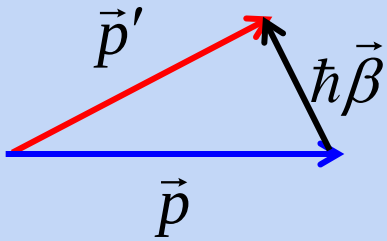
transition rate and scattering potential

$$S\left(\vec{p},\vec{p}'\right)=\frac{2\pi}{\hbar}\left|H_{p',p}\right|^2\delta\left(E'-E\right)$$

$$H_{p',p}=\frac{1}{\Omega}\int_{-\infty}^{+\infty}e^{-i\vec{p}'\bullet r/\hbar}U_S(r)e^{i\vec{p}\bullet r/\hbar}dr=\frac{1}{\Omega}\int_{-\infty}^{+\infty}U_S(r)e^{-i\left(\vec{p}'-\vec{p}\right)\bullet\vec{r}/\hbar}\,dr$$

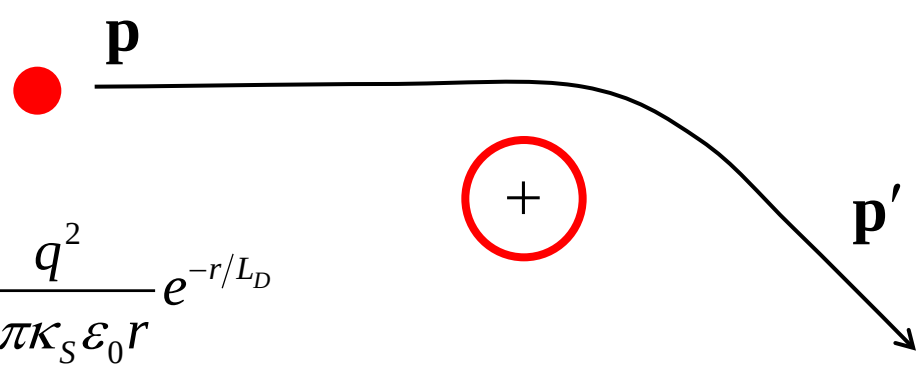
$$=\frac{1}{\Omega}\int_{-\infty}^{+\infty}U_S(r)e^{-i\vec{\beta}\bullet\vec{r}}\,d\vec{r}\equiv\frac{1}{\Omega}\tilde{U}_S\left(\vec{\beta}\right)$$

$$\vec{p}'=\vec{p}+\hbar\vec{\beta}$$



$$S\left(\vec{p},\vec{p}'\right)=\frac{2\pi}{\hbar}\frac{1}{\Omega^2}\left|\tilde{U}_S\left(\beta\right)\right|^2\delta\left(E'-E\right)$$

II scattering (Brooks-Herring)

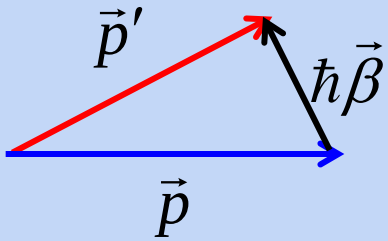


$$U_S(r) = \frac{q^2}{4\pi\kappa_S\varepsilon_0 r} e^{-r/L_D}$$

$$L_D = \sqrt{\frac{\kappa_S \varepsilon_0 k_B T}{q^2 n_0}} \quad \text{Debye length}$$

$$\tilde{U}_S(\beta) = \int_{-\infty}^{+\infty} U_S(r) e^{-i\vec{\beta}\bullet\vec{r}}\, d\vec{r}$$

$$\vec{p}' = \vec{p} + \hbar\vec{\beta}$$



Fourier transform of the screened Coulomb potential

$$\tilde{U}_S(\beta) = \int_{-\infty}^{+\infty} \frac{q^2}{4\pi\kappa_S\varepsilon_0 r} e^{-r/L_D} e^{-i\vec{\beta}\bullet\vec{r}}\, d\vec{r}$$

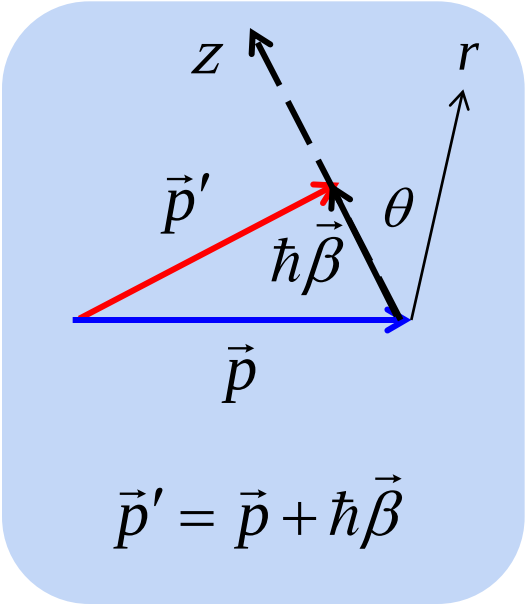
$$\tilde{U}_S(\beta) = \frac{q^2}{4\pi\kappa_S\varepsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} \int_0^{\infty} e^{-r/L_D} e^{-i\vec{\beta}\bullet\vec{r}}\, r\sin\theta d\theta dr$$

$$\vec{\beta}\bullet\vec{r} = \beta r\cos\theta$$

$$\sin\theta d\theta = -d(\cos\theta)$$

choose z-axis along *β:*

$$\tilde{U}_S(\beta) = \frac{q^2}{2\kappa_S\varepsilon_0} \int_0^{\infty} e^{-r/L_D} r dr \underbrace{\int_{-1}^{+1} e^{-i\beta r\cos\theta}\, d(\cos\theta)}_{\frac{2\sin(\beta r)}{\beta r}}$$



Fourier transform (ii)

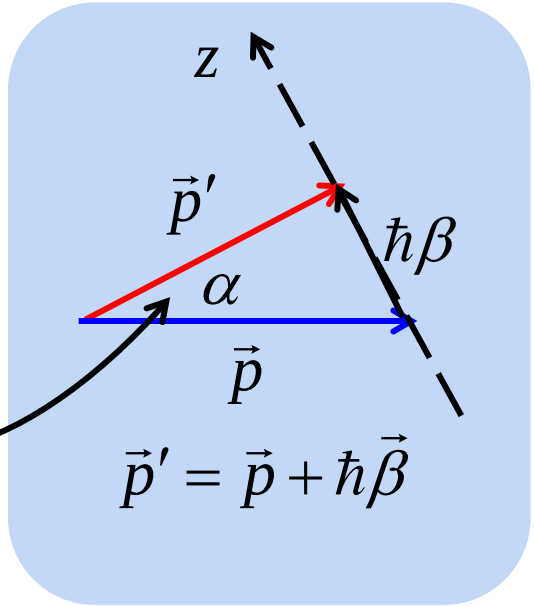
$$\tilde{U}_S(\beta) = \frac{q^2}{\kappa_S \varepsilon_0} \int_0^\infty \frac{e^{-r/L_D} \sin(\beta r)}{\beta} dr$$

$$\tilde{U}_S(\beta) = \frac{q^2}{\kappa_S \varepsilon_0} \left(\frac{1}{\beta^2 + 1/L_D^2} \right)$$

$$\tilde{U}_S(\beta) = \frac{q^2}{\kappa_S \varepsilon_0} \left(\frac{1}{4(p/\hbar)^2 \sin^2(\alpha/2) + 1/L_D^2} \right)$$

$$\vec{\beta} \bullet \vec{r} = \beta r \cos\theta$$

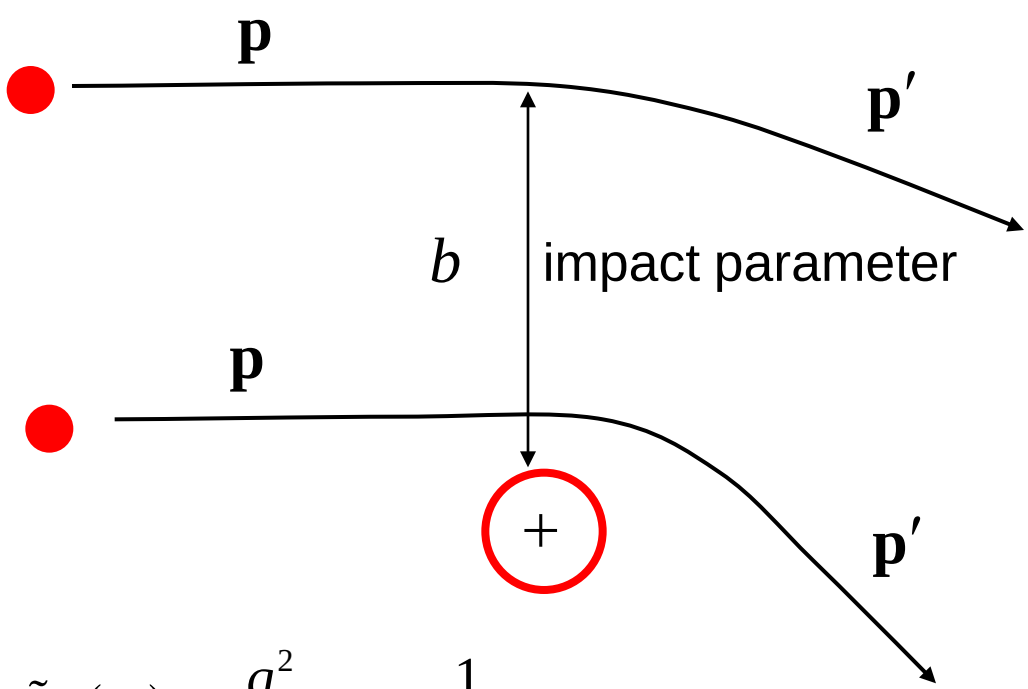
$$\sin\theta d\theta = -d(\cos\theta)$$



$$\hbar\beta = 2p\sin(\alpha/2)$$

small angle scattering preferred!!

small angle scattering



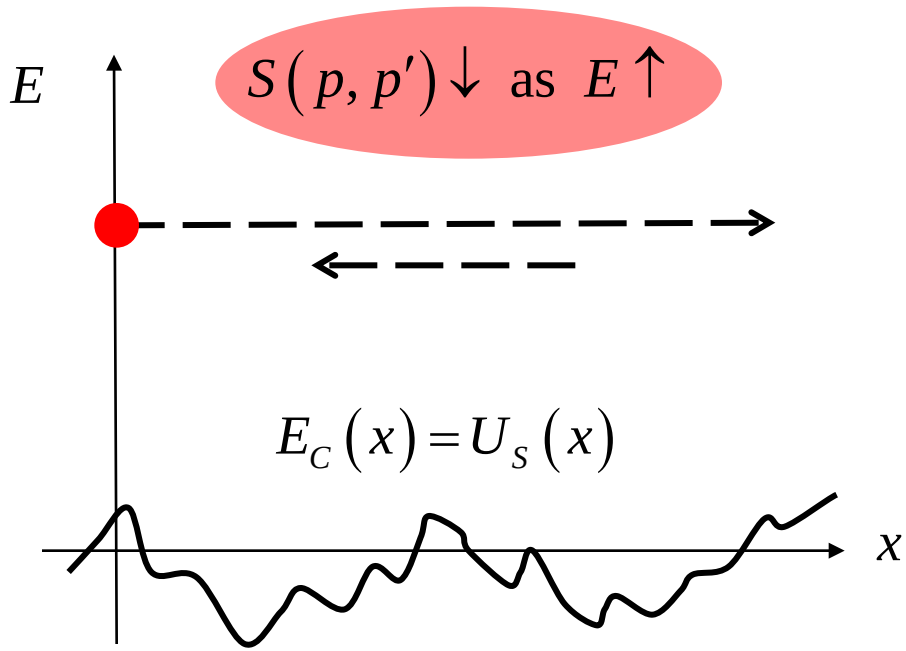
$$\tilde{U}_S(\beta) = \frac{q^2}{\kappa_S \varepsilon_0} \frac{1}{\left(\beta^2 + 1/L_D^2\right)}$$

$$U_S(r) = \frac{q^2}{4\pi\kappa_S \varepsilon_0 r} e^{-r/L_D}$$

II scattering of high energy carriers

$$\tilde{U}_S(\beta) = \frac{q^2}{\kappa_S \varepsilon_0}\left(\frac{1}{4(p/\hbar)^2 \sin^2(\alpha/2) + 1/L_D^2}\right)$$

For a given deflection angle, higher energies scatter less.



Random charges introduce random fluctuations in E_C, which act a scattering centers.

High energy electrons don't "see" these fluctuations and are not scattered as strongly.

II scattering: recap

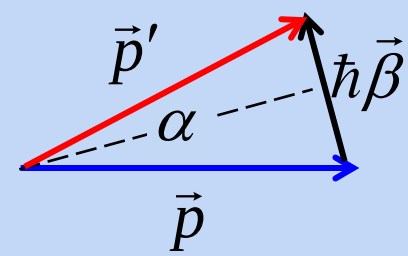
$$S(\vec{p},\vec{p}') = \frac{2\pi}{\hbar}\left|H_{p',p}\right|^2 \delta(E'-E) \quad H_{p,p'} = \frac{1}{\Omega}\tilde{U}_S(\beta) \quad \tilde{U}_S(\beta) = \frac{q^2}{\kappa_S \varepsilon_0}\frac{1}{\left(\beta^2 + 1/L_D^2\right)}$$

Need to multiple by the total number of ionized impurities in the volume, Ω.

$$S(\vec{p},\vec{p}') = \frac{2\pi q^4 N_I}{\hbar \kappa_S^2 \varepsilon_0^2 \Omega}\frac{\delta(E'-E)}{\left(\beta^2 + 1/L_D^2\right)^2}$$

$$S(\vec{p},\vec{p}') = \frac{2\pi q^4 N_I}{\hbar \kappa_S^2 \varepsilon_S^2 \Omega}\frac{\delta(E'-E)}{\left(\frac{4p^2}{\hbar^2}\sin^2\alpha/2 + 1/L_D^2\right)^2}$$

$$\vec{p}' = \vec{p} + \hbar\vec{\beta}$$



$$\frac{\hbar\beta}{2} = p\sin\alpha/2$$

examine result

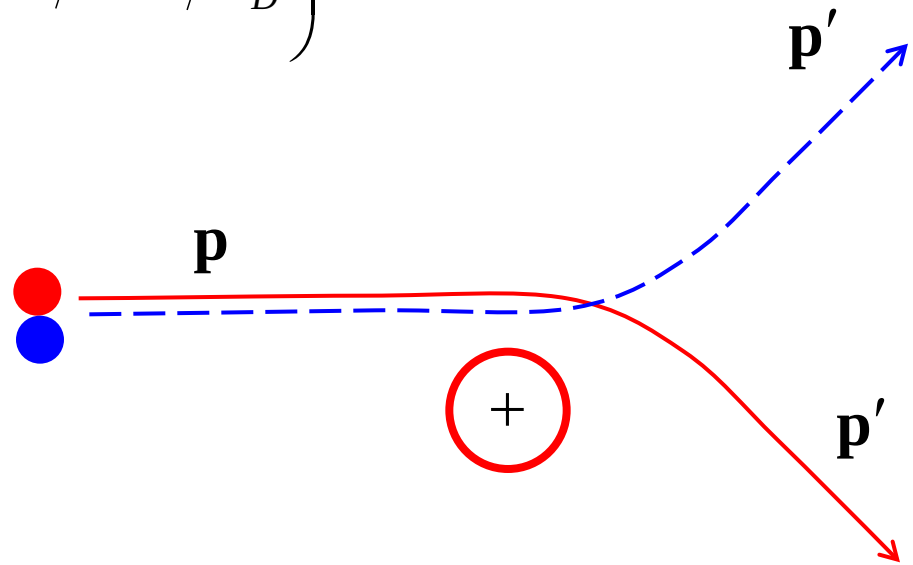
$$S\left(\vec{p},\vec{p}'\right)=\frac{2\pi q^4 N_I}{\hbar\kappa_S^2\varepsilon_0^2\Omega}\frac{\delta\left(E'-E\right)}{\left(\frac{4p^2}{\hbar^2}\sin^2\alpha/2+1/L_D^2\right)^2}$$

1) $S\left(\vec{p},\vec{p}'\right)\sim N_I$

2) $S\left(\vec{p},\vec{p}'\right)\sim q^4$

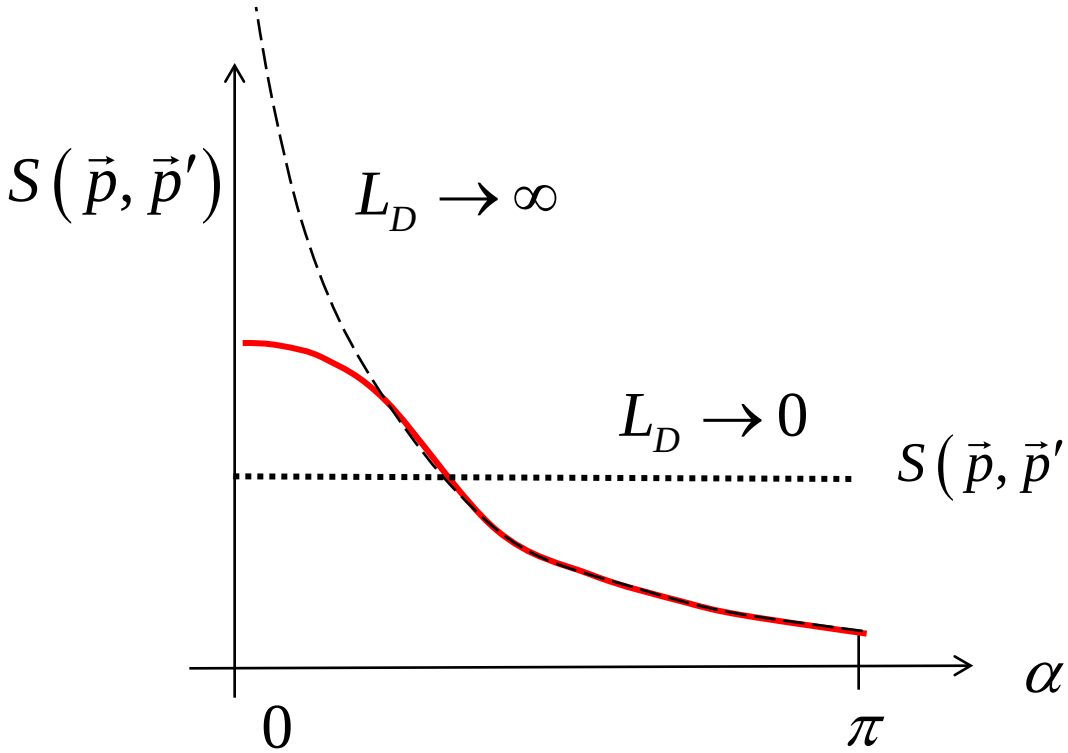
3) $S\left(\vec{p},\vec{p}'\right)\sim 1/E^2$

4) favors small angle scattering



examine result

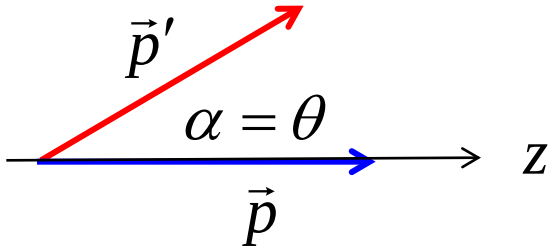
4) angular dependence $S(\vec{p},\vec{p}') = \frac{2\pi q^4 N_I}{\hbar \kappa_S^2 \varepsilon_0^2 \Omega} \frac{\delta(E'-E)}{\left(\frac{4p^2}{\hbar^2}\sin^2\alpha/2 + 1/L_D^2\right)^2}$



$$S(\vec{p},\vec{p}') \to \frac{2\pi q^4 N_I L_D^4}{\hbar \kappa_S^2 \varepsilon_0^2 \Omega} \delta(E'-E)$$

momentum relaxation time

$$\frac{1}{\tau_m} = \sum_{\vec{p}'} S\left(\vec{p}, \vec{p}'\right)\left(1 - \frac{p'}{p}\cos\alpha\right)$$



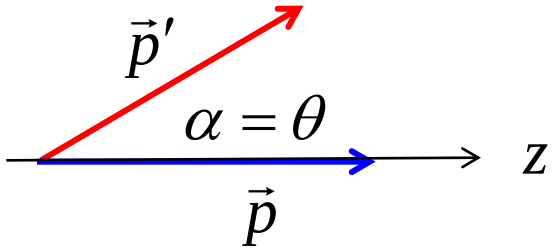
$$\frac{1}{\tau_m} = \sum_{\vec{p}'} S\left(\vec{p}, \vec{p}'\right)\left(1 - \cos\alpha\right)$$

$S\left(\vec{p}, \vec{p}'\right)$ favors small angles

expect: $1/\tau_m < 1/\tau \qquad \tau_m > \tau$

momentum relaxation time

$$\frac{1}{\tau_m} = \sum_{\vec{p}'} S\left(\vec{p}, \vec{p}'\right)\left(1 - \cos\alpha\right)$$



$$\tau_m(E) = \frac{16\sqrt{2m^*}\pi\kappa_S^2\varepsilon_0^2}{N_I q^4}\left[\ln\left(1+\gamma^2\right) - \frac{\gamma^2}{1+\gamma^2}\right]E^{3/2}$$

$$\gamma^2 = 8m^* EL_D^2/\hbar^2$$

See Lundstrom, pp. 69-70

$$\tau_m(E) \sim E^{3/2}$$

$$\tau_m(E) \approx \tau_0\left(E/k_B T_L\right)^{3/2} \qquad \tau_0 \sim T_L^{3/2} \qquad s = 3/2$$

to be continued in next lecture

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2) Screening
3) Brooks-Herring approach
4) **Conwell-Weisskopf approach**
5) Discussion
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(Reference: Chapter 2, Lundstrom, FCT)