Thermoelectricity: From Atoms to Systems

Week 4: Thermoelectric Systems
Tutorial 4.1 Homework solutions, problems 1-6

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Prob. 1. Heat balance equation

1-1. Second derivative of temperature

The heat generation rate per unit volume

$$\dot{q} = -\frac{dJ_Q}{dx} + J_e E$$

Coupled charge and heat current eq.

$$\boxed{\begin{aligned} J_Q &= STJ_e - \kappa \frac{dT}{dx} \\ E &= \frac{J_e}{\sigma} + S\frac{dT}{dx} \end{aligned}}$$

For one-dimensional problem, and assuming temperature- and position-independent properties,

$$\dot{q} = -\left[S\frac{dT}{dx}J_e - \kappa\frac{d^2T}{dx^2}\right] + \frac{J_e^{\,2}}{\sigma} + SJ_e\frac{dT}{dx}$$

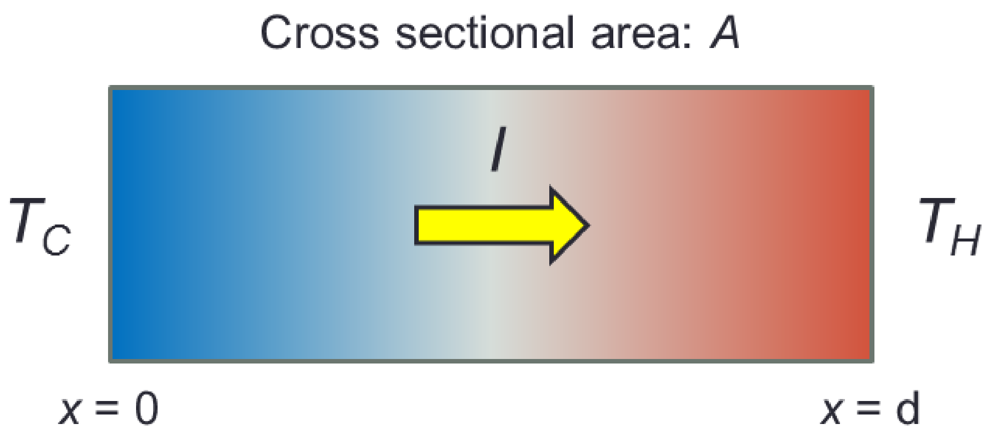
$$= \kappa\frac{d^2T}{dx^2} + \frac{J_e^{\,2}}{\sigma}$$

At steady-state, $\dot{q} = 0$

$$\therefore \frac{d^2T}{dx^2} = -\frac{1}{\kappa\sigma}J_e^{\,2}$$

Prob. 1. Heat balance equation

1-2. Temperature profile with current



$$\frac{d^2T}{dx^2} = -\frac{1}{\kappa\sigma} J_e^{\,2}$$

$$\Rightarrow \quad T(x) = -\frac{J_e^{\,2}}{2\kappa\sigma} x^2 + C_1 x + C_2$$

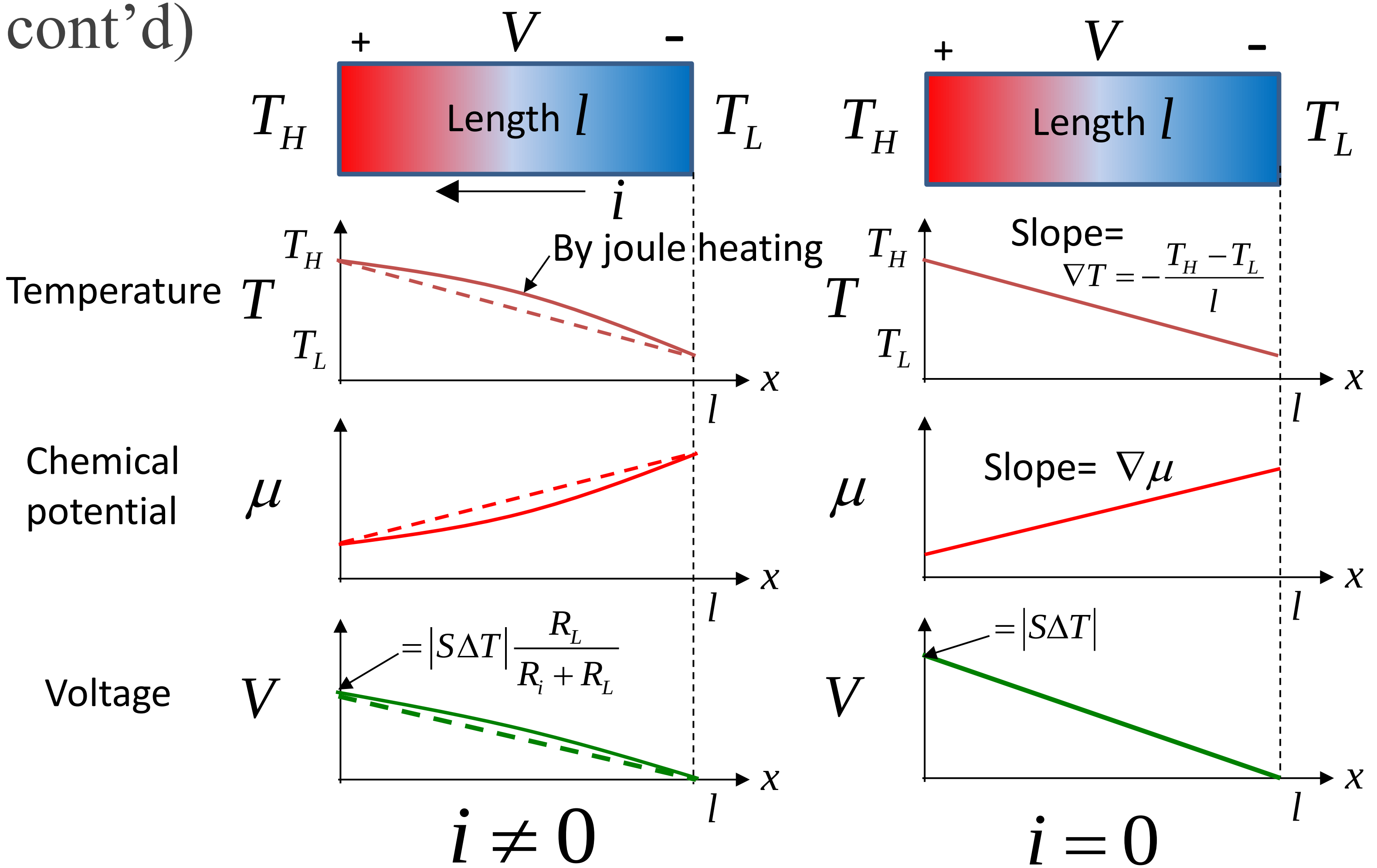
Boundary conditions:

$$\begin{cases} T(x=0) = T_C \\ T(x=d) = T_H \end{cases}$$

$$\therefore T(x) = -\frac{RI^2}{2Kd^2} x^2 + \left(\frac{\Delta T}{d} + \frac{RI^2}{2Kd} \right) x + T_C$$

Prob. 1. Heat balance equation

1-2. cont'd) Comparison between with and without current flow



Prob. 1. Heat balance equation

1-3. Heat balance equation at cold side

$$Q_{\text{out}} = \kappa A \left. \frac{dT}{dx} \right|_{x=0}$$

From 1-2.

$$T(x) = -\frac{RI^2}{2Kd^2}x^2 + \left(\frac{\Delta T}{d} + \frac{RI^2}{2Kd}\right)x + T_C$$

$$Q_{\text{out}} = \kappa A \left[-\frac{RI^2}{Kd^2}x + \left(\frac{\Delta T}{d} + \frac{RI^2}{2Kd}\right)\right]\Bigg|_{x=0}$$

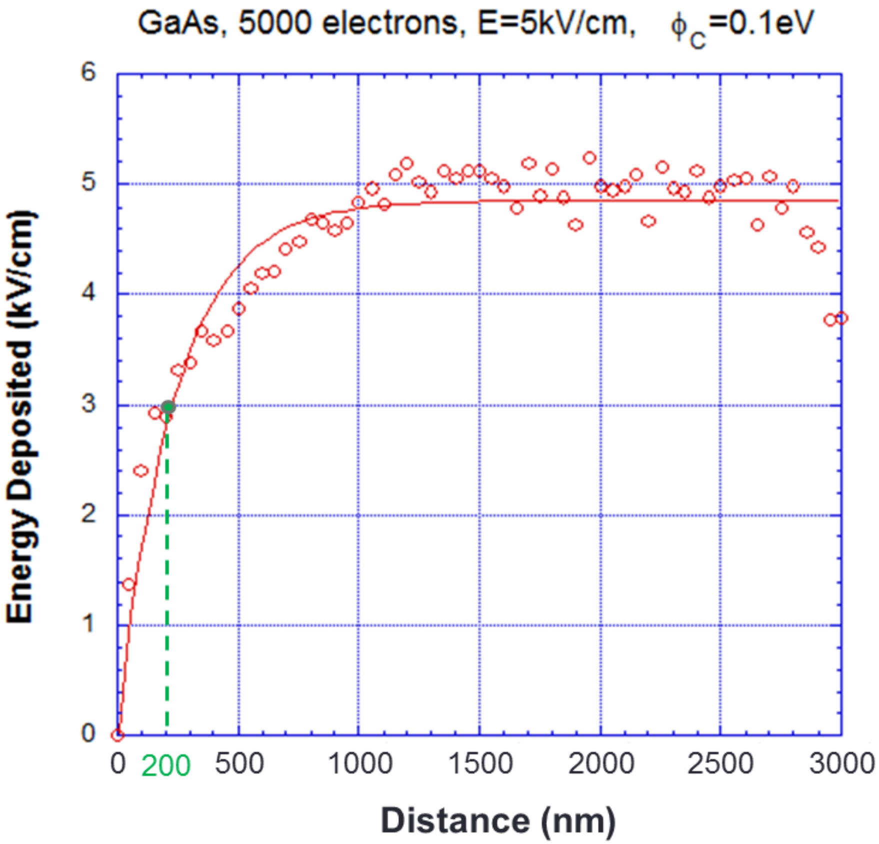
$$Q_{\text{out}} = \kappa A \left(\frac{\Delta T}{d} + \frac{RI^2}{2Kd}\right) = K\Delta T + \frac{1}{2}RI^2$$

$$\therefore\ Q_C = ST_C I - Q_{out} = \underline{ST_C I - \frac{1}{2}RI^2 - K\Delta T}$$

Prob. 2. Quasi-ballistic TE cooler

2-1. Temperature profile for quasi-ballistic TE cooler

$$\frac{dq_{\text{Joule}}}{dx} = \frac{1}{\sigma} J_e^{\,2}\left[1-\exp\left(-\frac{x}{\lambda_E}\right)\right] \qquad -\kappa\frac{d^2T}{dx^2} = \frac{dq_{\text{Joule}}}{dx}$$



$$\frac{d^2T}{dx^2} = -\frac{1}{\kappa\sigma} J_e^{\,2}\left[1-\exp\left(-\frac{x}{\lambda_E}\right)\right]$$

Boundary conditions:

$$\begin{cases} T(x=0) = T_C \\ T(x=d) = T_H \end{cases}$$

Prob. 2. Quasi-ballistic TE cooler

2-1. (cont'd)

$$\frac{d^2T}{dx^2} = -\frac{1}{\kappa\sigma} J_e^{\,2}\left[1 - \exp\left(-\frac{x}{\lambda_E}\right)\right]$$

$$T'' = -B\left[1 - \exp\left(-\frac{x}{\lambda_E}\right)\right]$$

where $B = \frac{1}{\kappa\sigma} J_e^{\,2} = \frac{RI^2}{Kd^2}$

$$T' = -B\lambda_E \exp\left(-\frac{x}{\lambda_E}\right) - Bx + C_1$$

$\Leftarrow$ $T(x=0) = T_C$

$$T = B\lambda_E^{\,2} \exp\left(-\frac{x}{\lambda_E}\right) - \frac{B}{2}x^2 + C_1 x + T_C - B\lambda_E^{\,2}$$

$\Leftarrow$ $T(x=d) = T_H$

$$T(x=d) = B\lambda_E^{\,2} \exp\left(-\frac{d}{\lambda_E}\right) - \frac{B}{2}d^2 + C_1 d + T_C - B\lambda_E^{\,2} = T_H$$

Prob. 2. Quasi-ballistic TE cooler

2-1. (cont'd)

$$B\lambda_E^{\ 2}\exp\left(-\frac{d}{\lambda_E}\right)-\frac{B}{2}d^2+C_1 d+T_C-B\lambda_E^{\ 2}=T_H$$

$$C_1=-\frac{B\lambda_E^{\ 2}}{d}\exp\left(-\frac{d}{\lambda_E}\right)+\frac{B}{2}d+\frac{T_H-T_C}{d}+\frac{B\lambda_E^{\ 2}}{d}$$

$$C_1=\frac{B\lambda_E^{\ 2}}{d}\left[1-\exp\left(-\frac{d}{\lambda_E}\right)\right]+\frac{B}{2}d+\frac{\Delta T}{d}$$

$$\therefore T=B\lambda_E^{\ 2}\exp\left(-\frac{x}{\lambda_E}\right)-\frac{B}{2}x^2+\left\{\frac{B\lambda_E^{\ 2}}{d}\left[1-\exp\left(-\frac{d}{\lambda_E}\right)\right]+\frac{B}{2}d+\frac{\Delta T}{d}\right\}x+T_C-B\lambda_E^{\ 2}$$

$$B=\frac{RI^2}{Kd^2}$$

$$\therefore T(x)=-\frac{RI^2}{2Kd^2}x^2+\left\{\frac{\Delta T}{d}+\frac{RI^2}{2Kd}+\frac{RI^2\lambda_E^{\ 2}}{Kd^3}\left[1-\exp\left(-\frac{d}{\lambda_E}\right)\right]\right\}x+T_C-\frac{RI^2\lambda_E^{\ 2}}{Kd^2}\left[1-\exp\left(-\frac{x}{\lambda_E}\right)\right]$$

Prob. 2. Quasi-ballistic TE cooler

2-2. Heat dissipation at cold side

$$Q_{\text{out}} = \kappa A \left.\frac{dT}{dx}\right|_{x=0}$$

$$\frac{dT(x)}{dx} = -\frac{RI^2}{Kd^2}x + \left\{\frac{\Delta T}{d} + \frac{RI^2}{2Kd} + \frac{RI^2\lambda_E{}^2}{Kd^3}\left[1-\exp\left(-\frac{d}{\lambda_E}\right)\right]\right\} - \frac{RI^2\lambda_E}{Kd^2}\exp\left(-\frac{x}{\lambda_E}\right)$$

$$\left.\frac{dT(x)}{dx}\right|_{x=0} = \left\{\frac{\Delta T}{d} + \frac{RI^2}{2Kd} + \frac{RI^2\lambda_E{}^2}{Kd^3}\left[1-\exp\left(-\frac{d}{\lambda_E}\right)\right]\right\} - \frac{RI^2\lambda_E}{Kd^2}$$

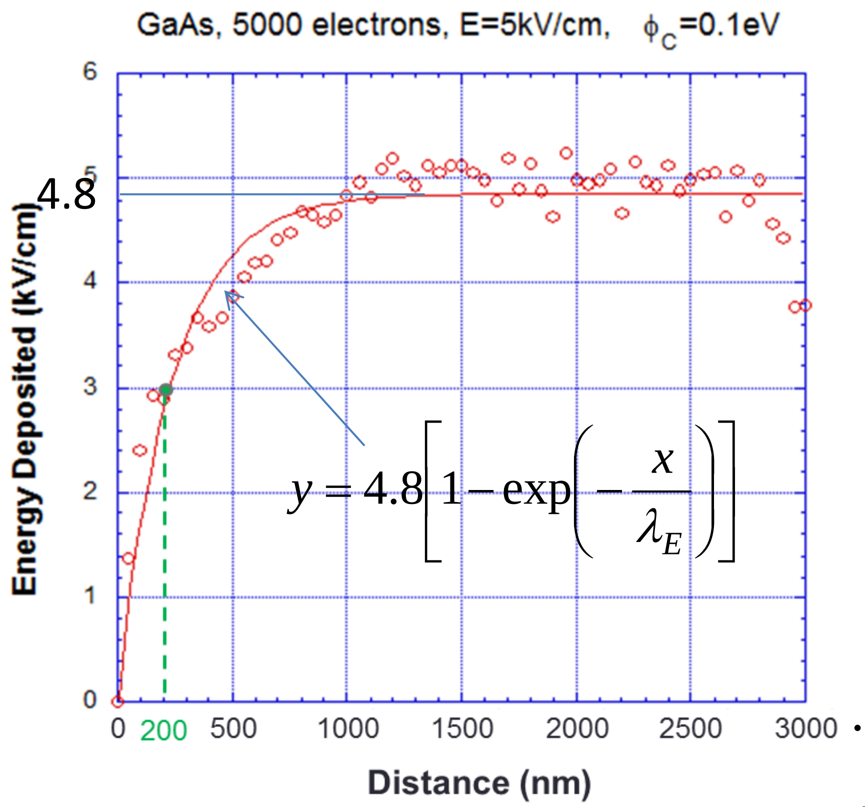
$$\kappa A\left.\frac{dT(x)}{dx}\right|_{x=0} = \left\{K\Delta T + \frac{RI^2}{2} + \frac{RI^2\lambda_E{}^2}{d^2}\left[1-\exp\left(-\frac{d}{\lambda_E}\right)\right]\right\} - \frac{RI^2\lambda_E}{d}$$

$$\therefore Q_{\text{out}} = RI^2\left\{\frac{1}{2} - \frac{\lambda_E}{d} + \left(\frac{\lambda_E}{d}\right)^2\left[1-\exp\left(-\frac{d}{\lambda_E}\right)\right]\right\} + K\Delta T$$

Prob. 2. Quasi-ballistic TE cooler

2-3. Fraction of Joule heating dissipated to cathode

$$Q_{\text{out}} = RI^2 \underbrace{\left\{ \frac{1}{2} - \frac{\lambda_E}{d} + \left(\frac{\lambda_E}{d}\right)^2 \left[1 - \exp\left(-\frac{d}{\lambda_E}\right) \right] \right\}}_{\text{Fraction of } RI^2} + K\Delta T$$



$$d = 3000 \text{ nm}$$

From the left figure, at $x = \lambda_E$

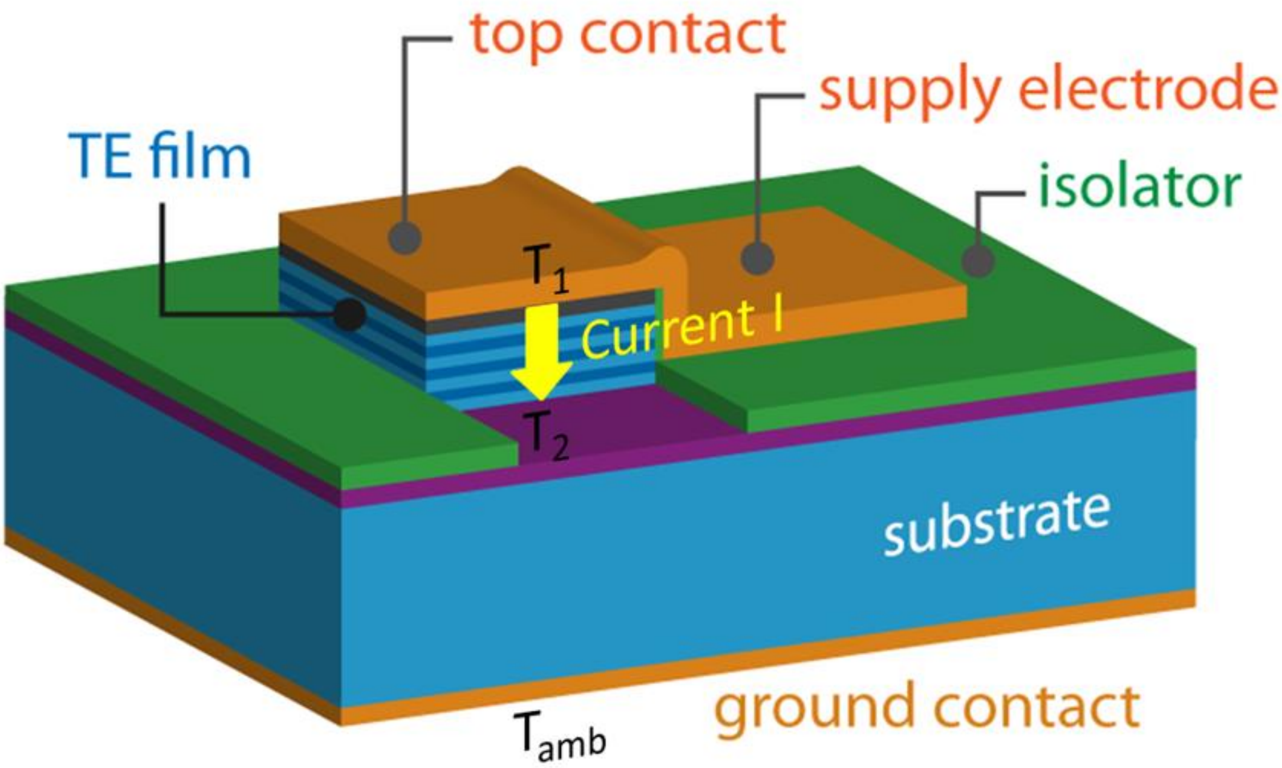
$$y = 4.8[1 - \exp(-1)] = 3.0$$

$$\therefore \lambda_E = 200 \text{ nm}$$

$$\frac{\lambda_E}{d} = \frac{1}{15}$$

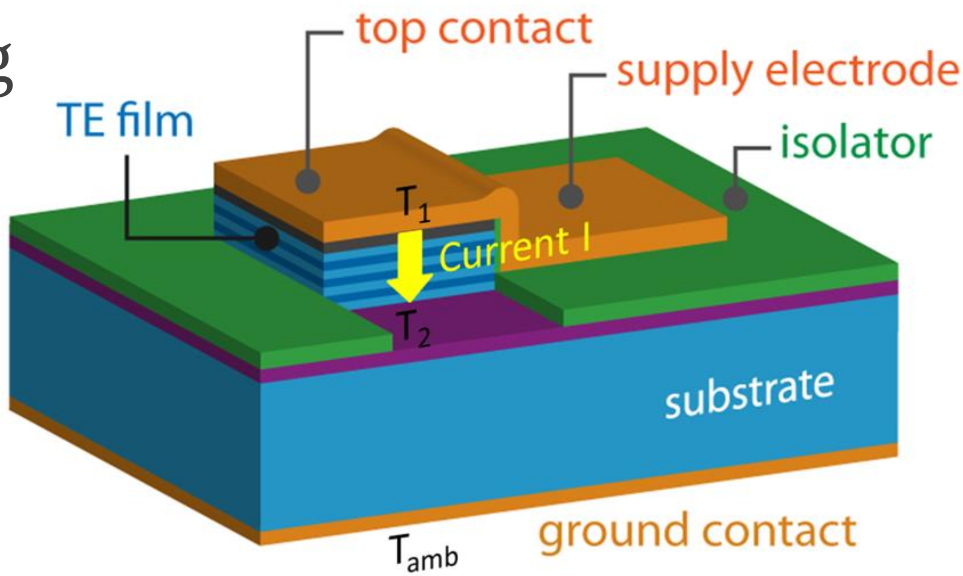
$$\therefore \frac{1}{2} - \frac{\lambda_E}{d} + \left(\frac{\lambda_E}{d}\right)^2 \left[1 - \exp\left(-\frac{d}{\lambda_E}\right) \right] = \underline{0.44}$$

Prob. 3. Thermoelectric cooler



Prob. 3. Thermoelectric cooler

3-1. Max cooling



$$0 = ST_C I - \frac{1}{2} RI^2 - K\Delta T \quad \Rightarrow \quad \Delta T = -\frac{R}{2K} I^2 + \frac{ST_C}{K} I$$

$$\Delta T = -\frac{R}{2K}\left(I - \frac{ST_C}{R}\right)^2 + \frac{1}{2} ZT_C{}^2$$

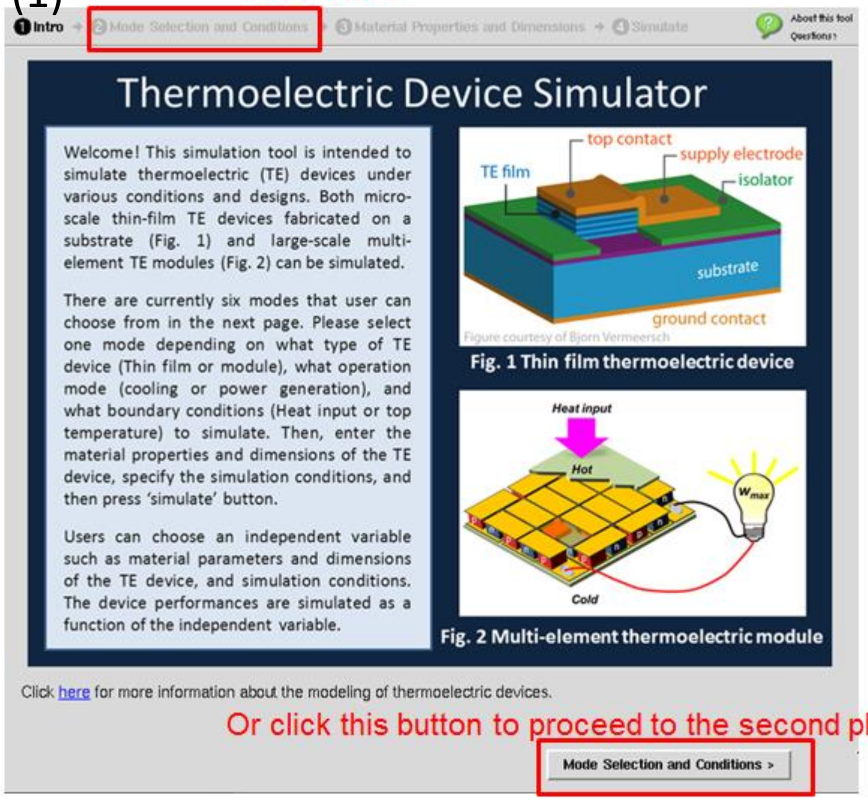
$$\therefore \Delta T_{max} = \frac{1}{2} ZT_C{}^2 \quad \text{when} \quad I = \frac{ST_C}{R}$$

Prob. 3. Thermoelectric cooler

3-2. Simulation (https:nanohub.org/tools/thermo)

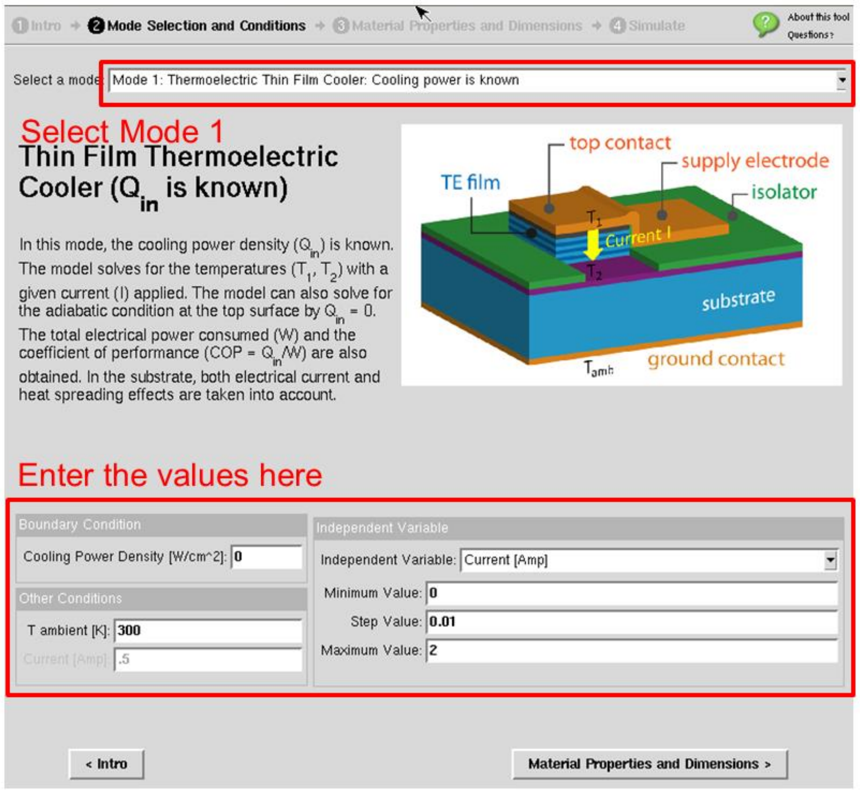
(1)

Click here



Or click this button to proceed to the second phase

(2)

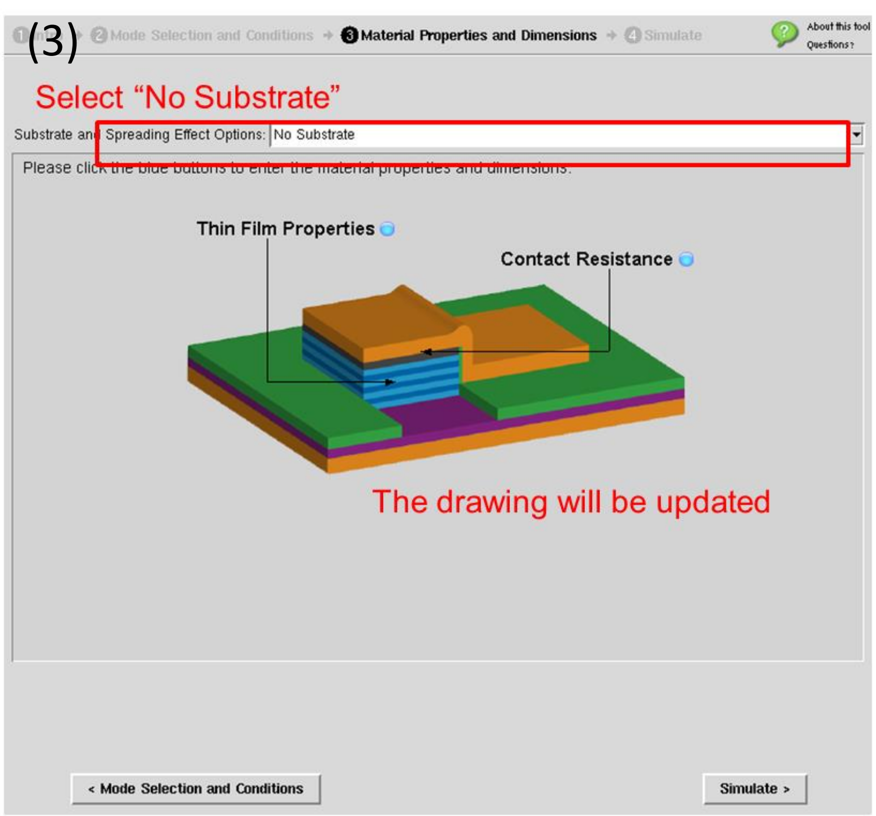


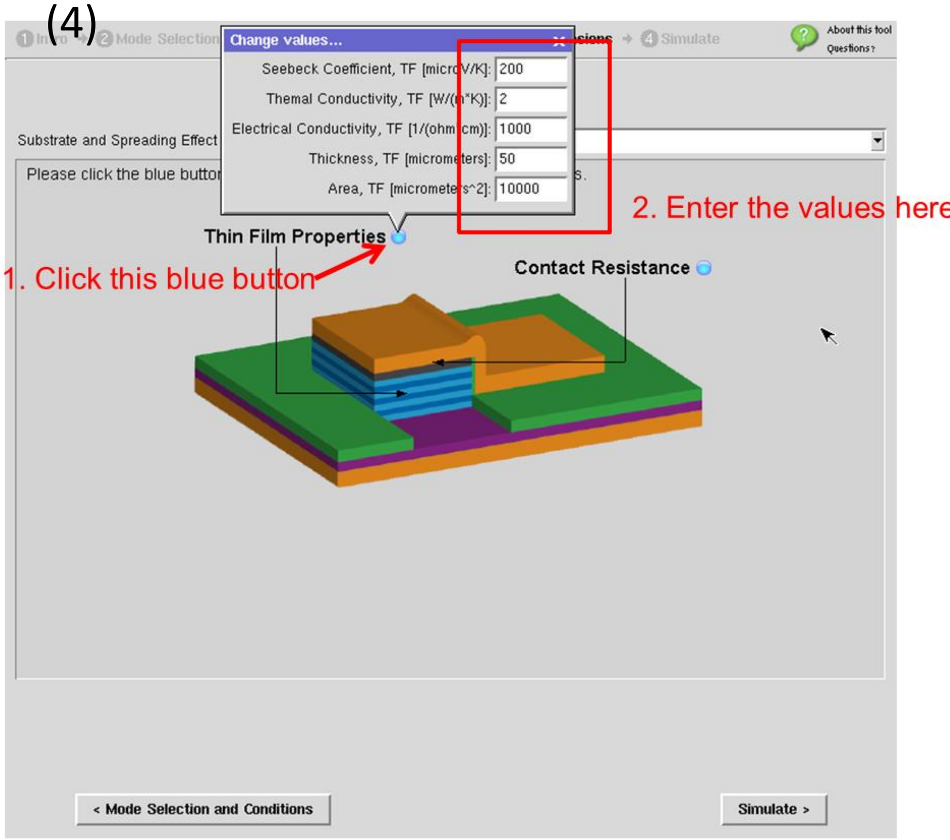
Select Mode 1

Enter the values here

Prob. 3. Thermoelectric cooler

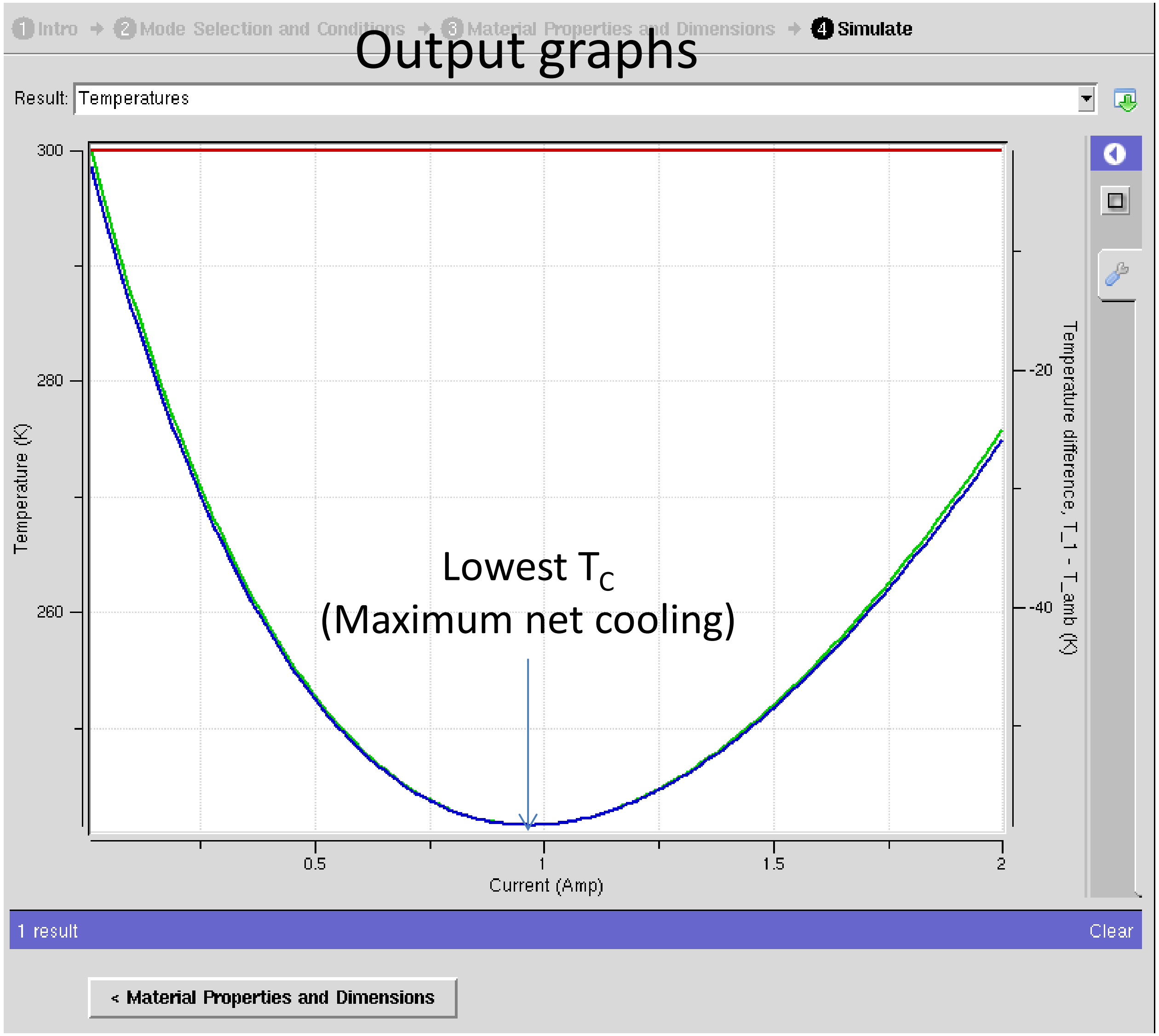
3-2. Simulation for maximum net cooling (cont'd)





Prob. 3. Thermoelectric cooler

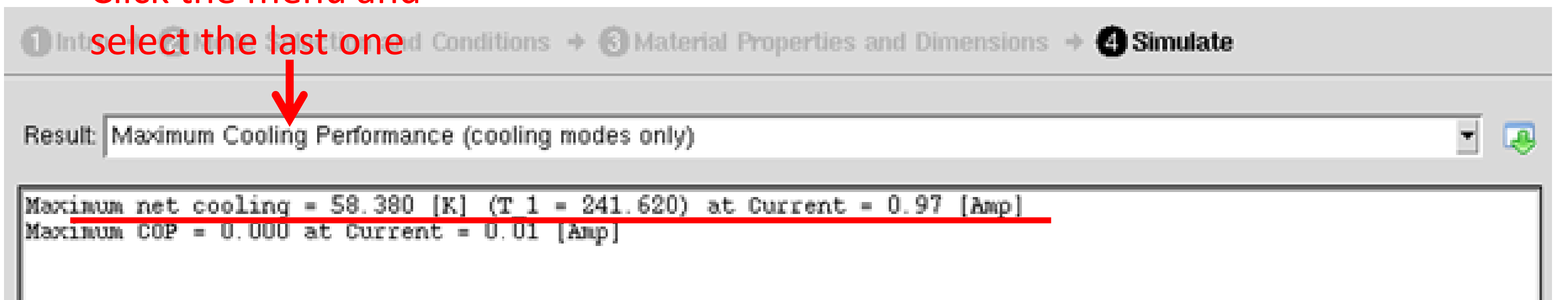
3-2. (cont'd)



Prob. 3. Thermoelectric cooler

3-2. (cont'd)

Click the menu and select the last one



$$\Delta T_{\max} = 58.4\ \mathrm{K},\ I_{\mathrm{opt}} = 0.97\ \mathrm{A}$$

Check: $T_C = 241.6\ \mathrm{K}$

$$Z = \frac{(200\times10^{-6}\ \mathrm{V/K})^2(1000\times10^2\ \Omega^{-1}\mathrm{m}^{-1})}{(2\ \mathrm{W/mK})} = 0.002\ \mathrm{K}^{-1}$$

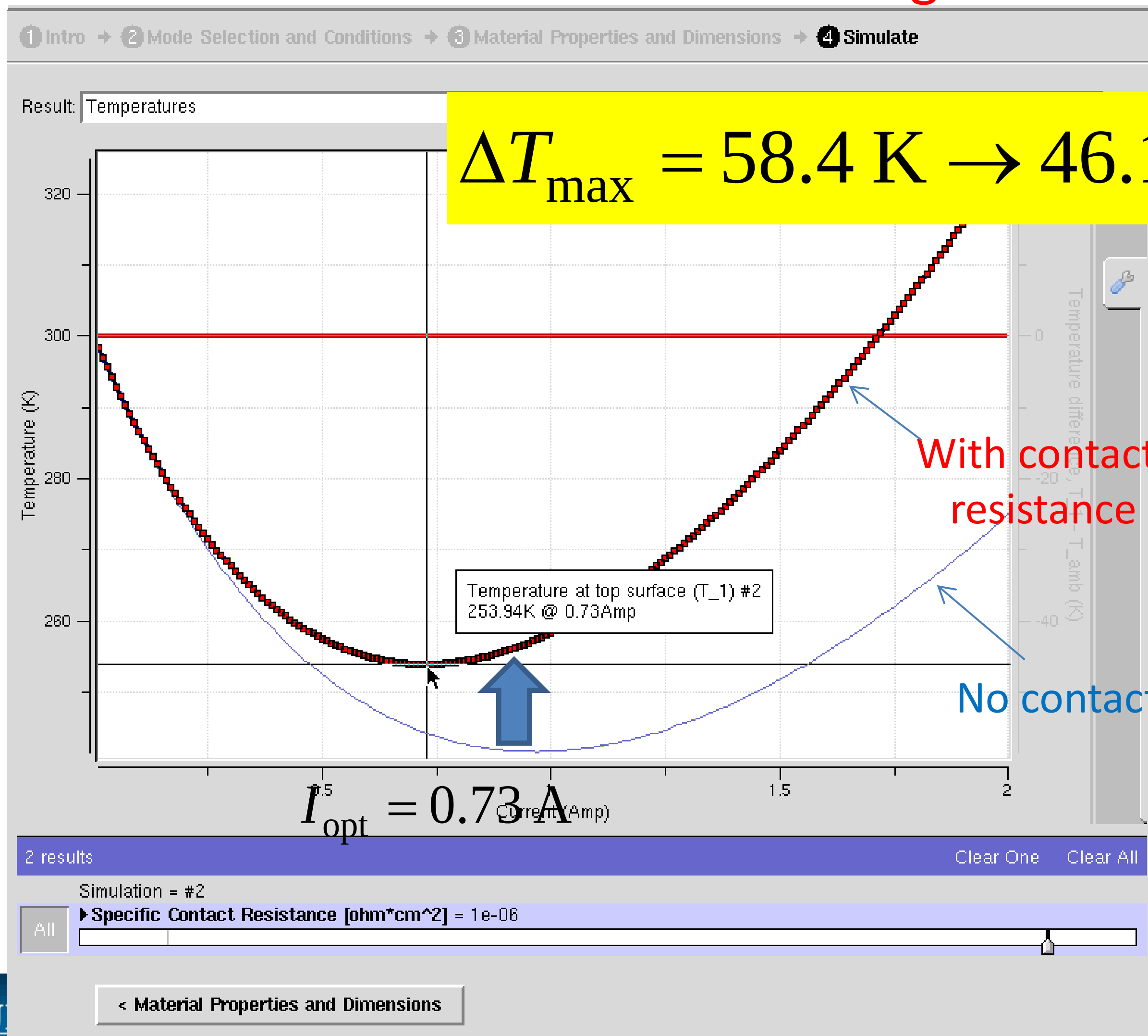
$$\therefore\ \frac{1}{2}ZT_C{}^2 = \frac{1}{2}(0.002\ \mathrm{K}^{-1})(241.6\ \mathrm{K})^2 = 58.4\ \mathrm{K}$$

$$= \Delta T_{\max}$$ (match!)

Prob. 3. Thermoelectric cooler

3-3. With specific contact resistance= 1e-6 Ω cm^2

-> Additional Joule heating



$$\Delta T_{\max} = 58.4\ \mathrm{K} \rightarrow 46.1\ \mathrm{K}$$

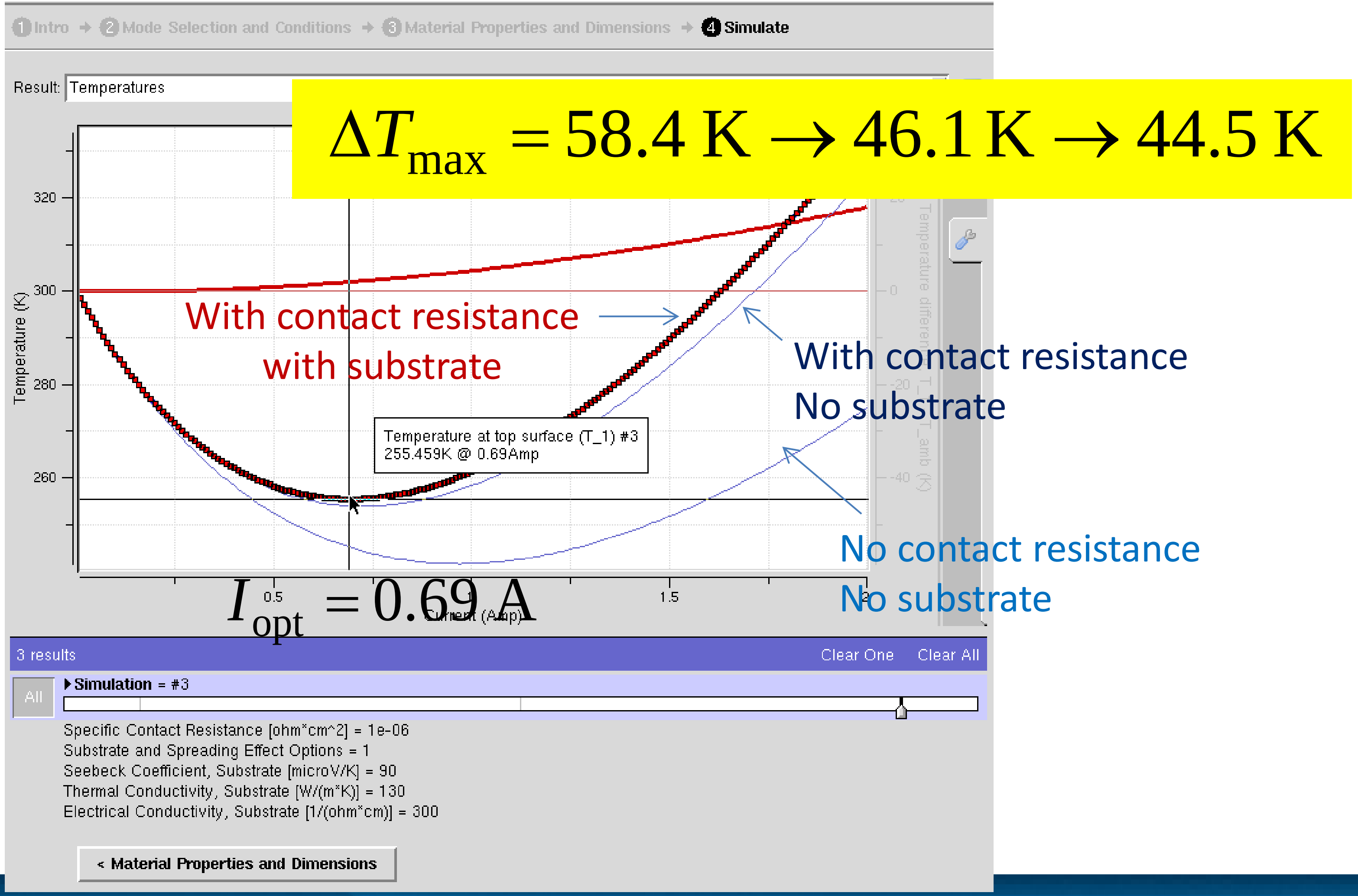
With contact resistance

No contact resistance

$$I_{\mathrm{opt}} = 0.73\ \mathrm{A}$$

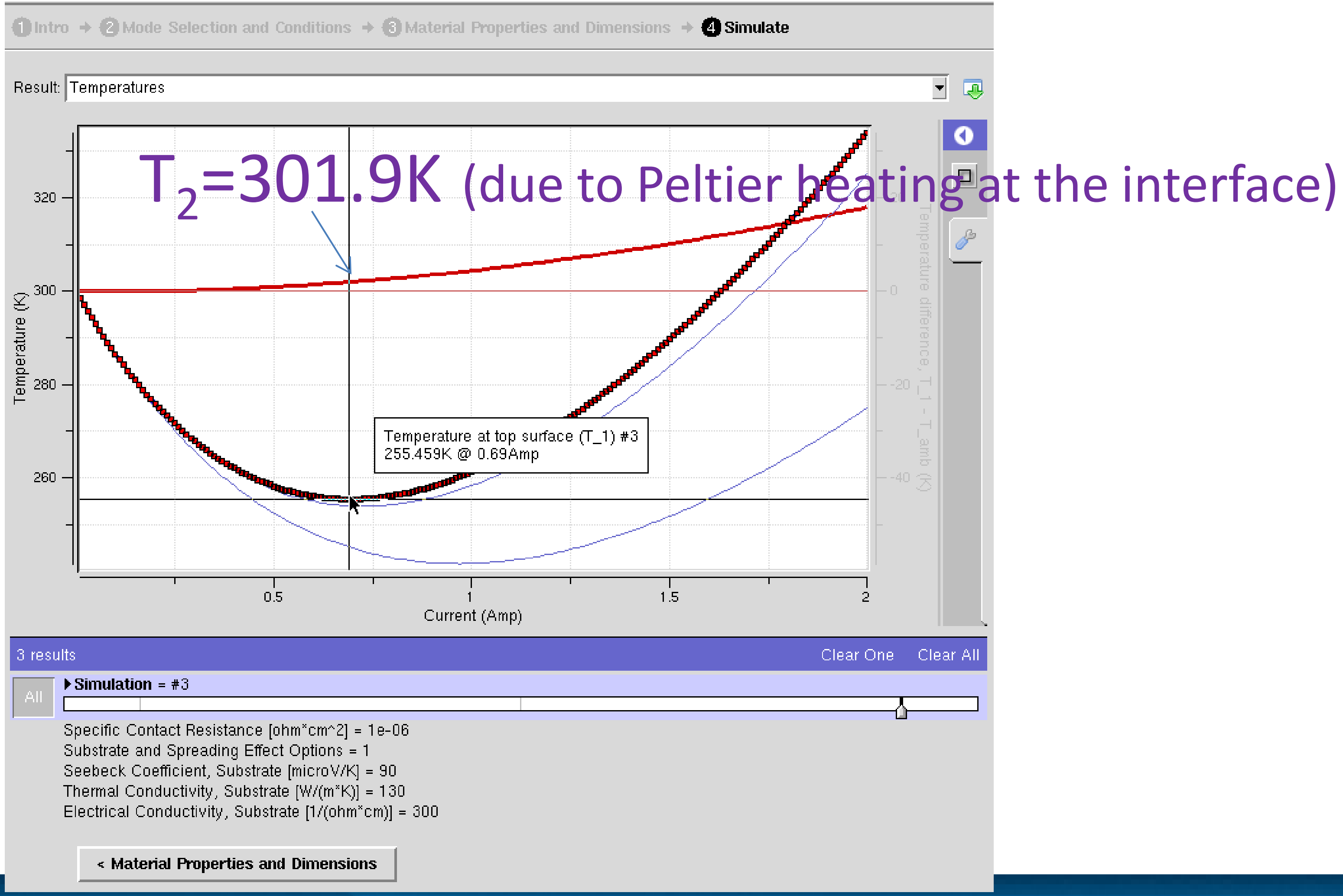
Prob. 3. Thermoelectric cooler

3-4. With substrate with spreading effect



Prob. 3. Thermoelectric cooler

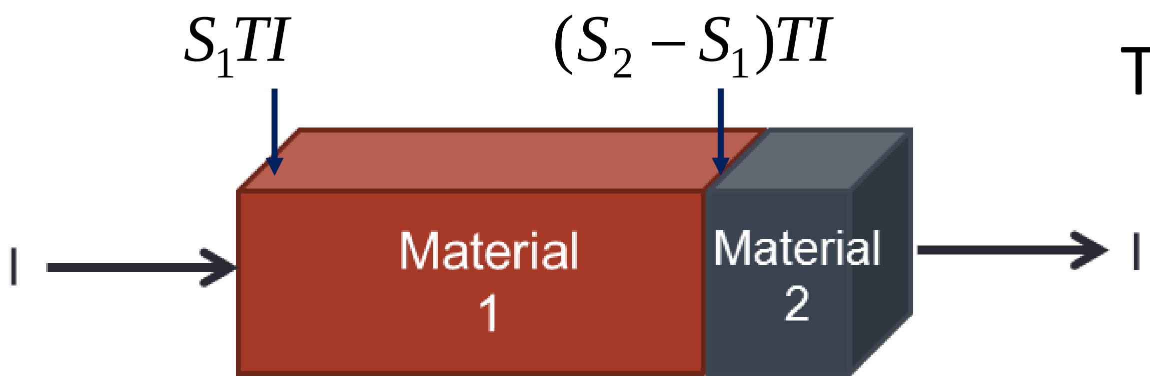
3-5. With substrate with spreading effect on T_2



Prob. 4. Segmented TE element

In a single material: $\Delta T_{max} = \frac{1}{2} Z T_C{}^2$ when $I = \frac{S T_C}{R}$ or $I^2 R = S T_C I$

(Joule)=(Peltier)



L_1	$L_2 = L_1/4$
S_1	$S_2 = 2S_1$
σ_1	$\sigma_2 = \sigma_1/4$
R_1	$R_2 = R_1$

To keep this condition

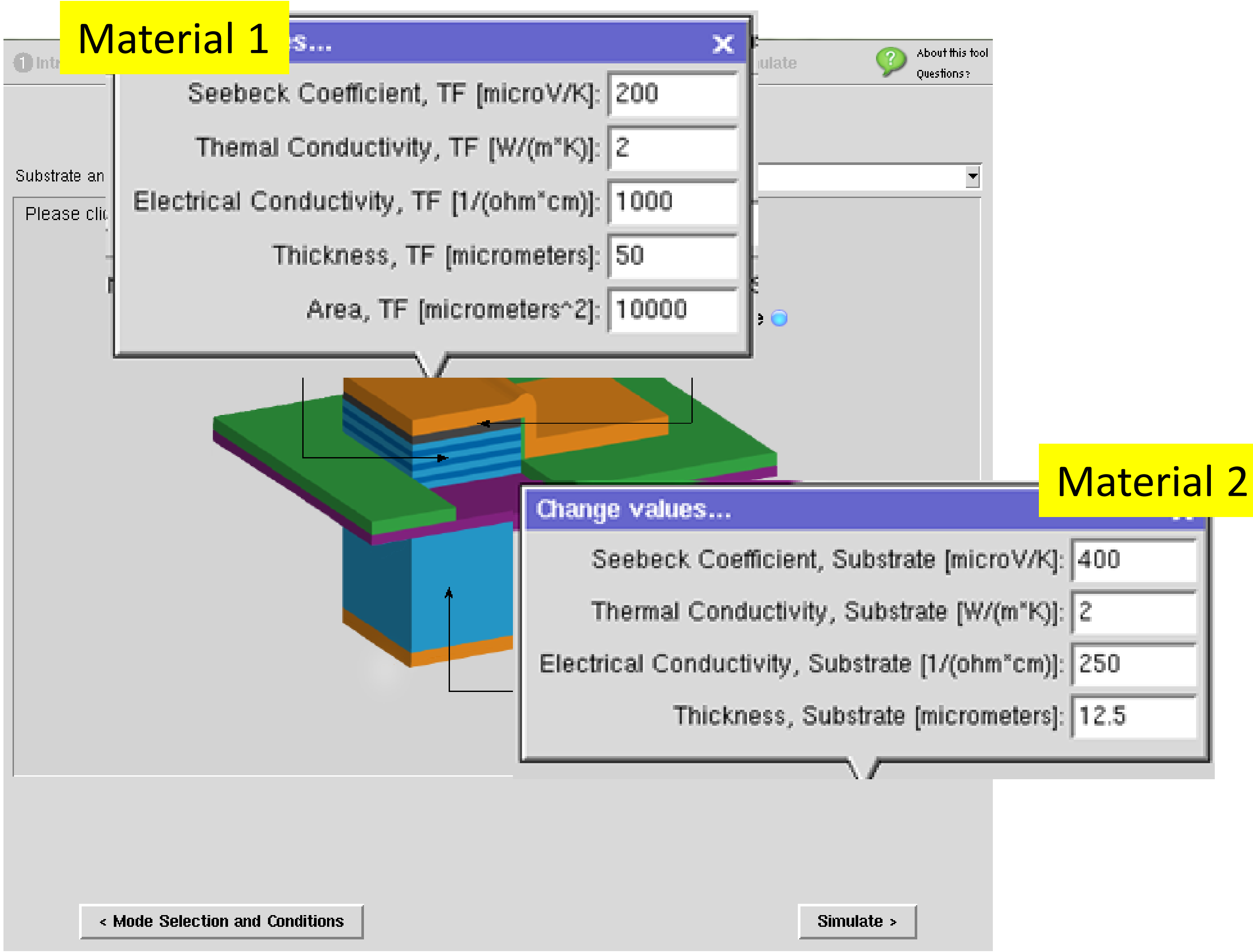
$$S_1 TI = (S_2 - S_1)TI$$

$$\therefore S_2 = 2S_1$$

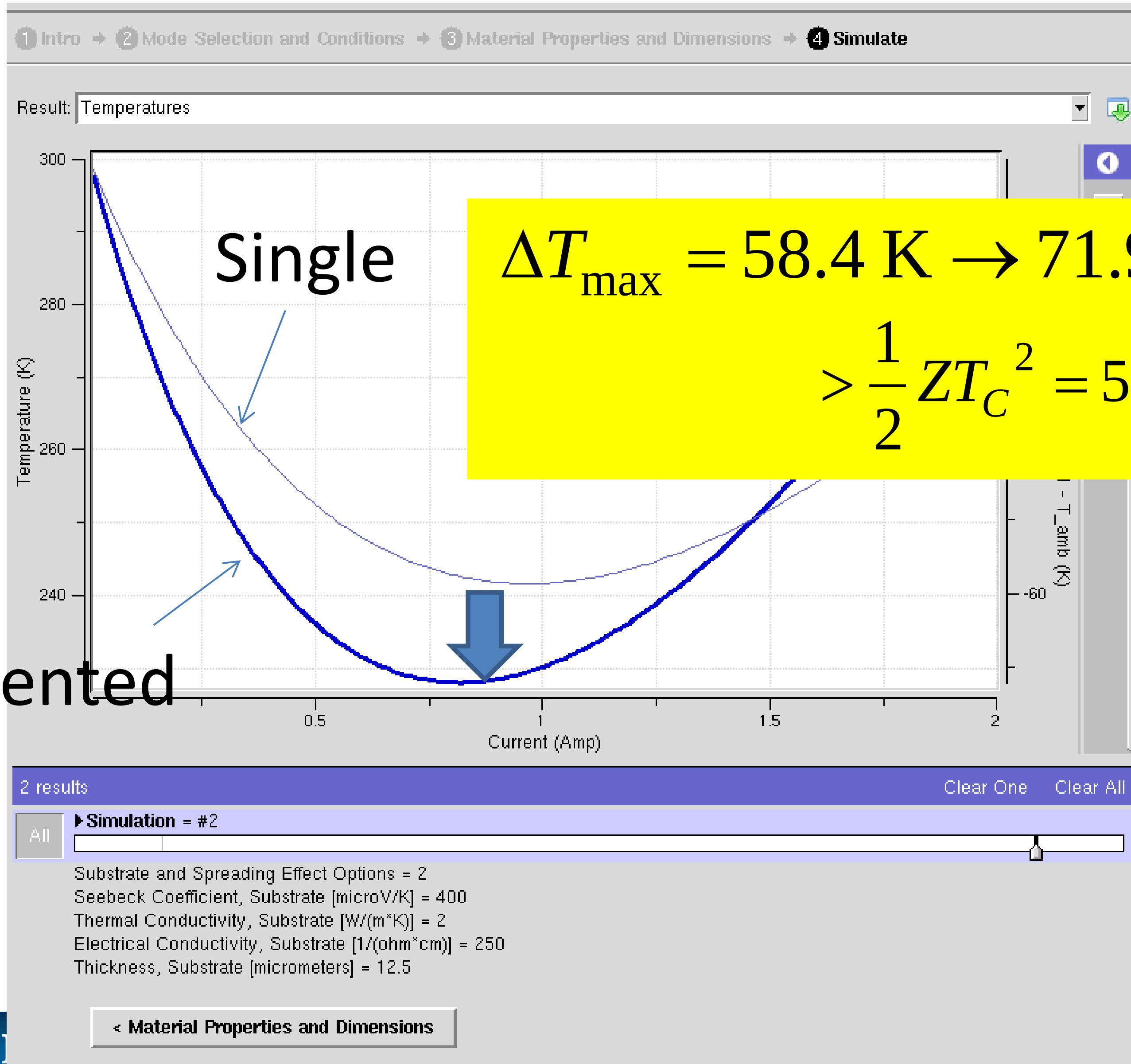
$$R_1 = R_2$$

Also, $S_1{}^2 \sigma_1 = S_2{}^2 \sigma_2$

Prob. 4. Segmented TE element



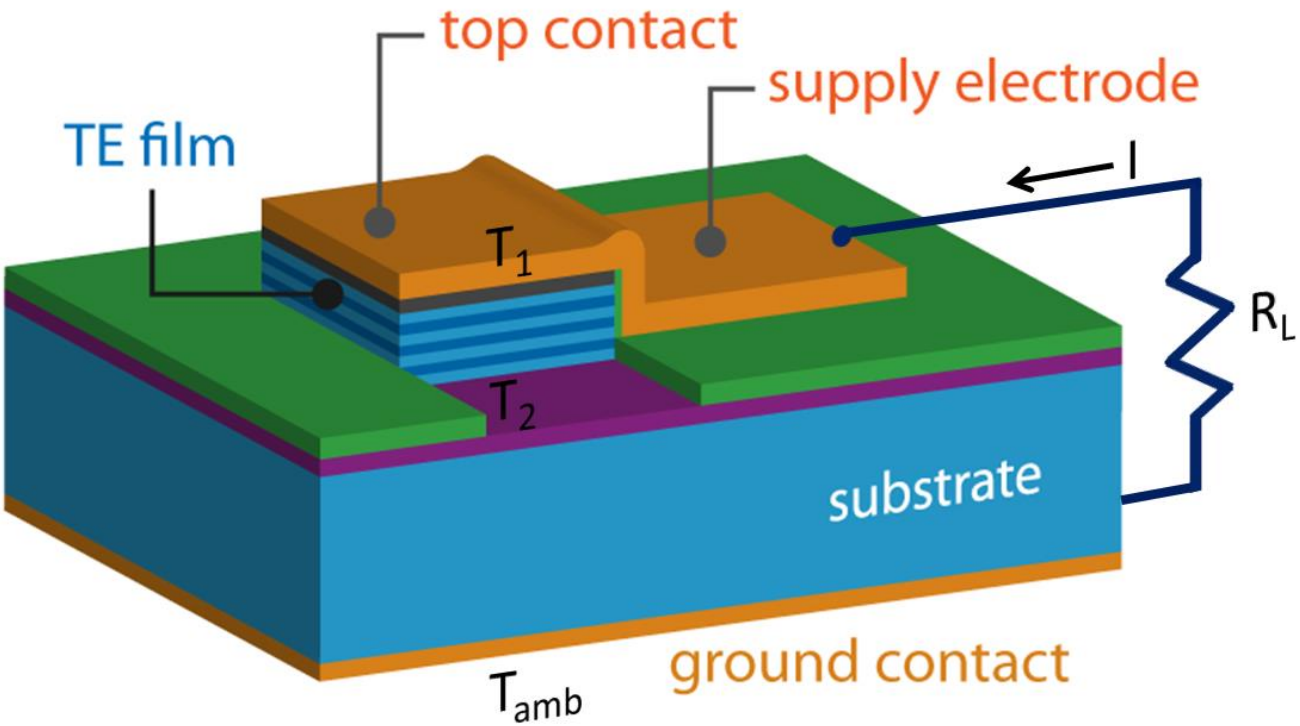
Prob. 4. Segmented TE element



$$\Delta T_{\max} = 58.4\ \mathrm{K} \rightarrow 71.97\ \mathrm{K}$$

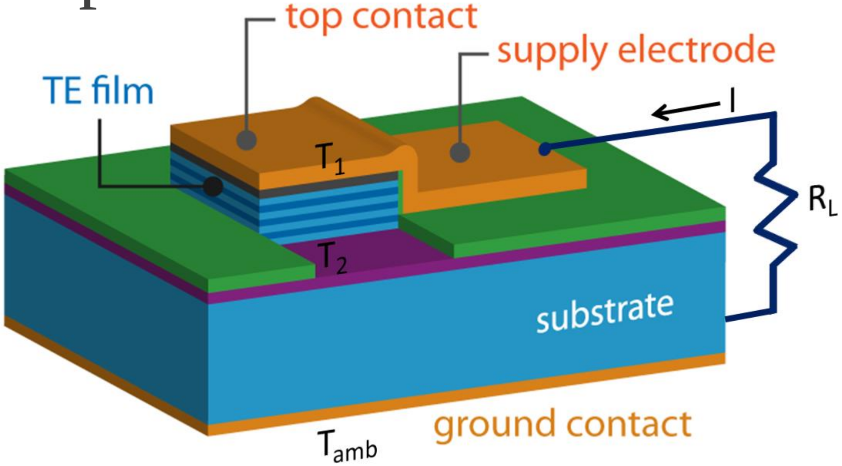
$$> \frac{1}{2} Z T_C^{\ 2} = 52.0\ \mathrm{K}$$

Prob. 5. TE power generator

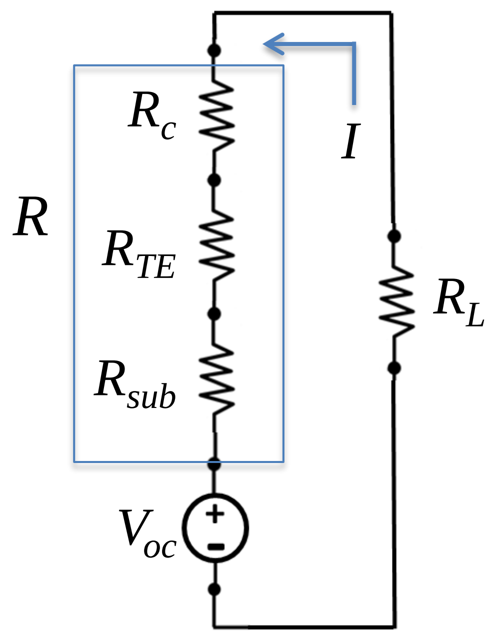


Prob. 5. TE power generator

5-1. Power output



< Electrical resistance network for power generation>



Seebeck effect: $V_{oc}=S_{\mathrm{TE}}(T_1\text{-}T_2)+S_{\mathrm{sub}}(T_2\text{-}T_{\mathrm{amb}})$

$V_{oc}=S_{\mathrm{TE}}\Delta T$ (without substrate)

Ohmic law: $V_{oc}=I(R+R_L) = S_{\mathrm{TE}}\Delta T$

$$\therefore I = \frac{S(T_H - T_C)}{R+R_L},\ P_{\mathrm{out}} = \frac{S^2(\Delta T)^2 R_L}{(R+R_L)^2}$$

Prob. 5. TE power generator

5-2. Maximum power output

$$P_{\text{out}} = \frac{S^2(\Delta T)^2 R_L}{(R+R_L)^2}$$

Differentiate P_{out} with respect to R_L

$$\frac{dP_{\text{out}}}{dR_L} = \frac{S^2(\Delta T)^2\left[(R+R_L)^2 - 2R_L(R+R_L)\right]}{(R+R_L)^4} = 0$$

$$\therefore\ R_L = R$$

$$P_{\text{out,max}} = \frac{S^2(\Delta T)^2}{4R}$$

Prob. 5. TE power generator

5-3. Maximum efficiency

$$\eta = \frac{I\left(S\Delta T - IR\right)}{ST_H I - \frac{1}{2}RI^2 + K\Delta T} \qquad \text{(a)}$$

Differentiate η with respect to *I*, and $\frac{d\eta}{dI} = 0$

$$(S\Delta T - 2RI)(K\Delta T + SIT_H - 1/2I^2R) - I(S\Delta T - RI)(ST_H - RI) = 0$$

$$I = \frac{K\Delta T}{ST_M}(\gamma - 1) = \frac{S\Delta T}{ZT_M R}(\gamma - 1)$$

$$= \frac{S\Delta T}{R}\frac{(\gamma - 1)}{(\gamma^2 - 1)} = \boxed{\frac{S\Delta T}{R(\gamma + 1)}} \qquad \text{(b)}$$

where $\gamma = \sqrt{1 + ZT_M}$

$$ZT_M = \gamma^2 - 1$$

Prob. 5. TE power generator

5-3. Maximum efficiency (cont'd)

Plug (b) into (a) to get

$$\therefore \eta_{\max} = \frac{(\gamma - 1)}{\left[(\gamma + 1) - \frac{\Delta T}{T_H}\right]} \frac{\Delta T}{T_H}$$

From $I = \frac{S\Delta T}{R(\gamma + 1)} = \frac{S\Delta T}{R + R_L}$

$$\therefore R_L = \gamma R$$

5-4. (Carnot limit)

$$\eta_{\max} = \frac{(\gamma - 1)}{\left[(\gamma + 1) - \frac{\Delta T}{T_H}\right]} \frac{\Delta T}{T_H} = \frac{\Delta T}{T_H}$$

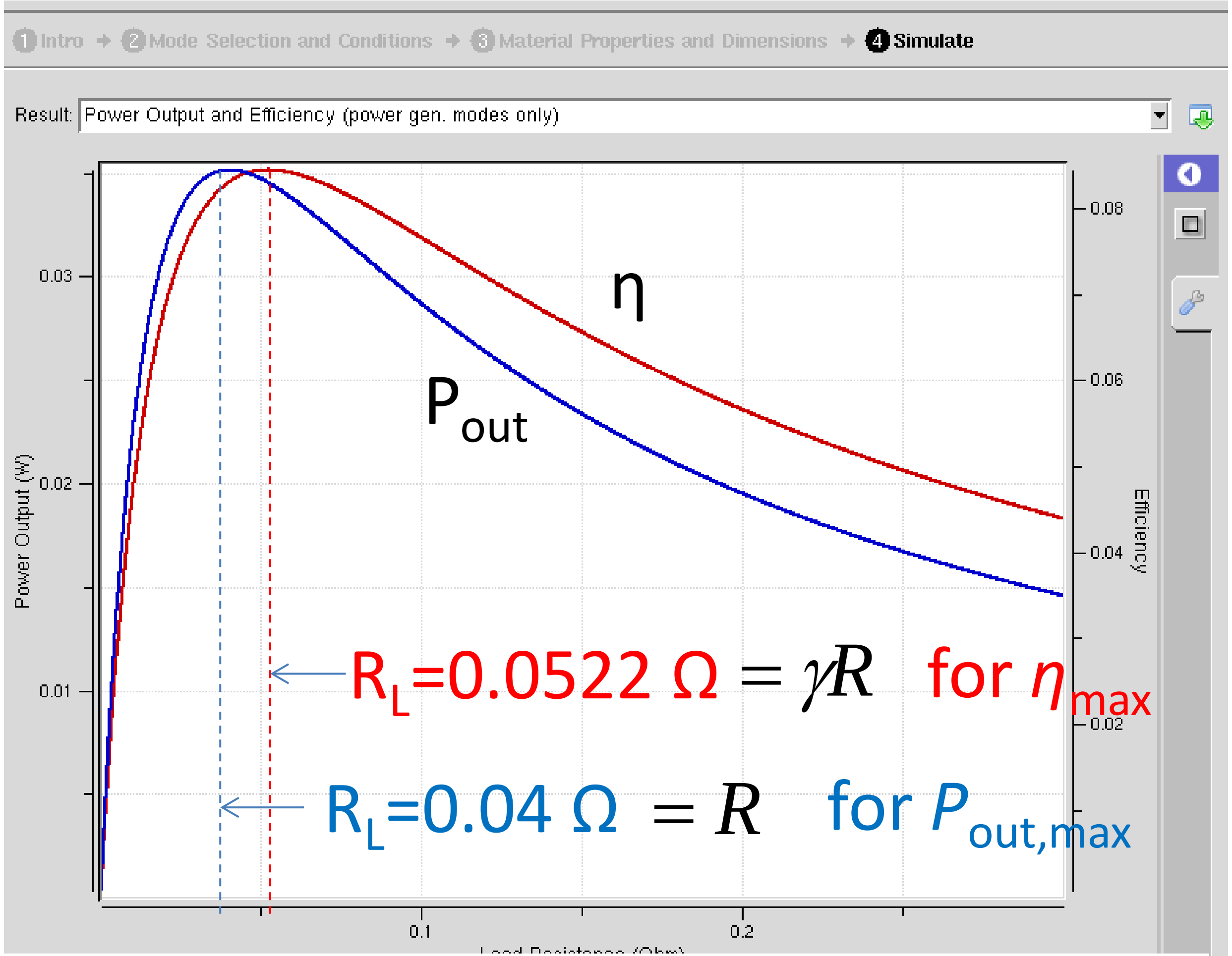
$$(\gamma - 1) = (\gamma + 1) - \frac{\Delta T}{T_H}$$

True only if $\gamma = \infty$

or $\boxed{ZT_M = \infty}$

Prob. 5. TE power generator

5-5 & 5-6 Simulations



$$Z = \frac{(250\times10^{-6}\ \text{V/K})^2(50000\ \Omega^{-1}\text{m}^{-1})}{(2\ \text{W/mK})} = .001563$$

$$T_M = 450\ \text{K}$$

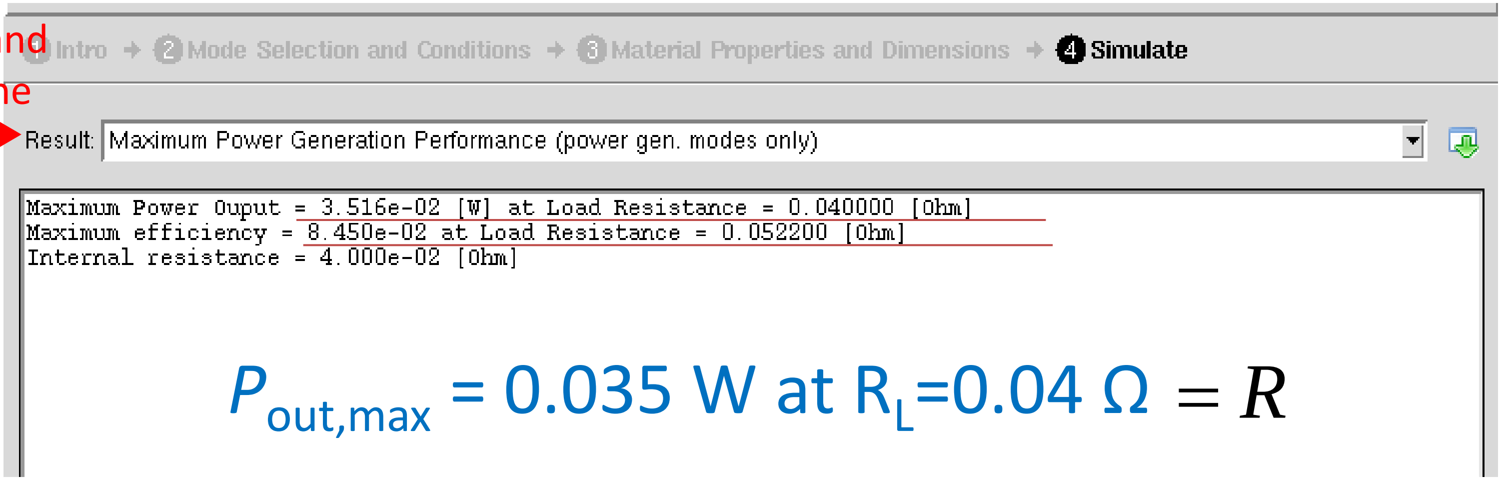
$$\therefore ZT_M = 0.70$$

$$\gamma = \sqrt{1+0.70} = 1.305$$

Prob. 5. TE power generator

5-5 & 5-6 (cont'd)

Click the menu and select the last one



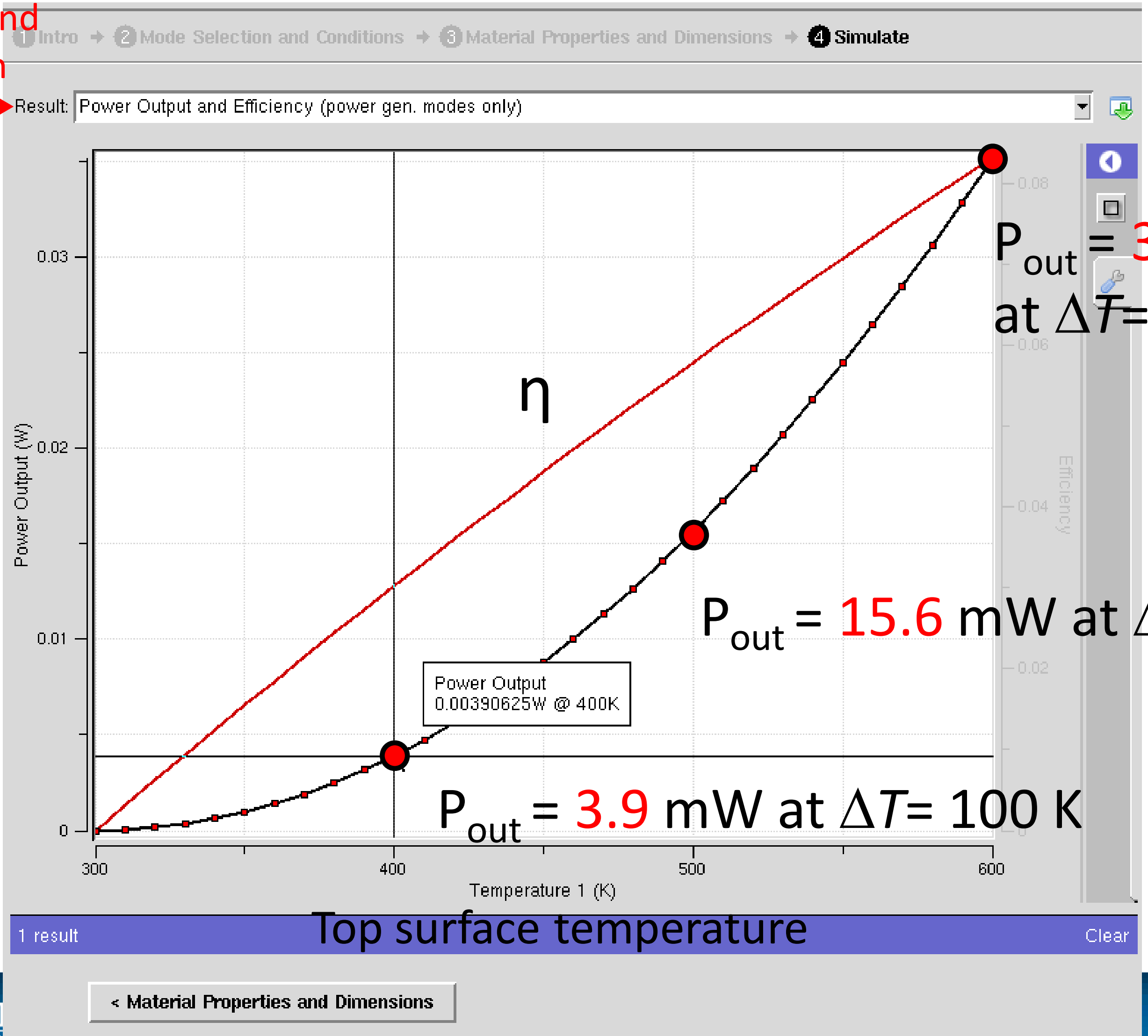
$P_{out,max}$ = 0.035 W at R_L=0.04 Ω $= R$

η_{max} = 0.085 at R_L=0.0522 Ω $= \gamma R$

Prob. 5. TE power generator

5-7. Power output as a function of ΔT

Click the menu and select this option

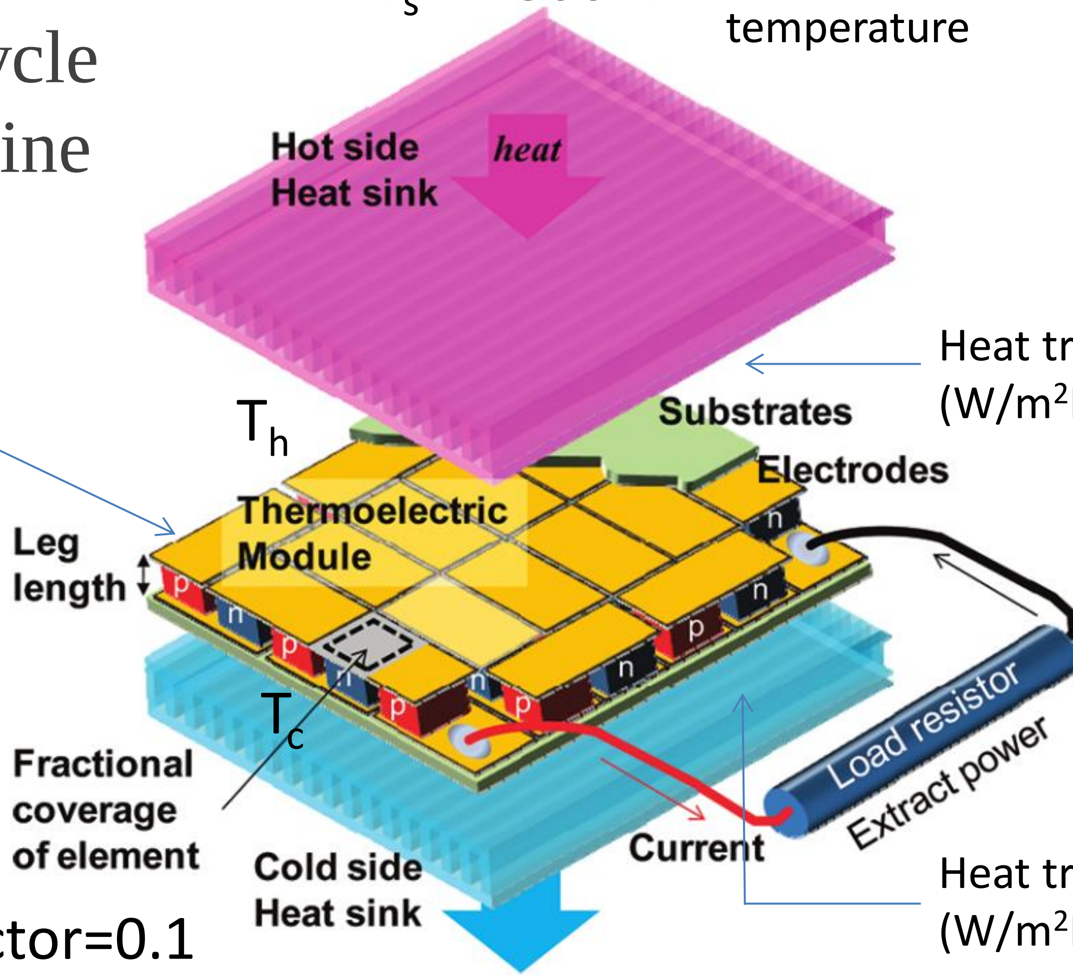


Prob. 6. Thermoelectric system

* An example of TE topping cycle on steam turbine

T_s = 1800 K

Heat source (combustion) temperature



Heat transfer coefficient (W/m^2K) at hot side

1000 x 1000

elements per m^2

Fill factor=0.1

Heat transfer coefficient (W/m^2K) at cold side

T_g = 800 K

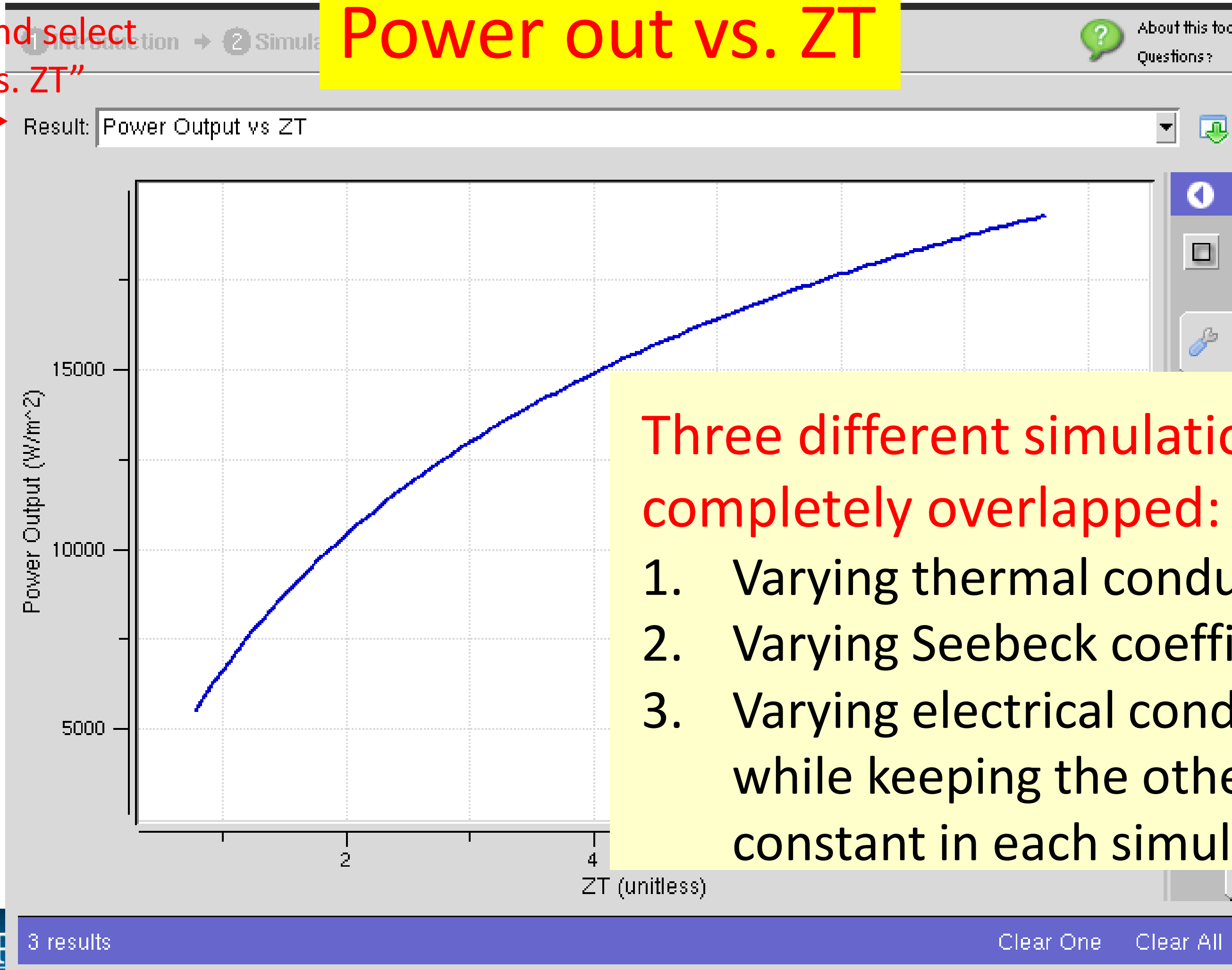
Gas temperature in the turbine

Prob. 6. Thermoelectric system

6-1. Power output as a function of material properties and heat transfer coefficients

Click the menu and select "Power Output vs. ZT"

Power out vs. ZT



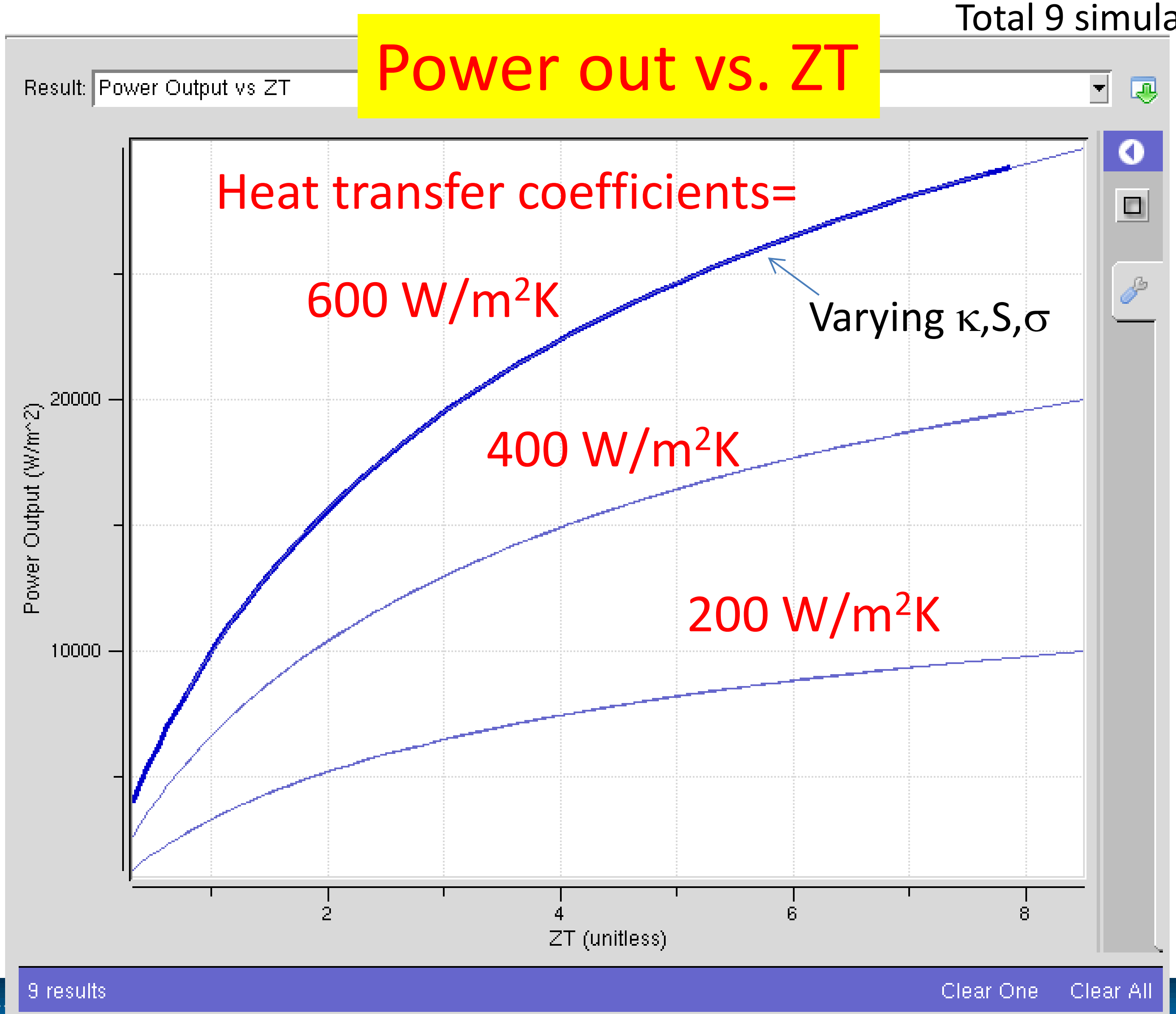
Three different simulations completely overlapped:

1. Varying thermal conductivity (κ)
2. Varying Seebeck coefficient (S)
3. Varying electrical conductivity (σ) while keeping the other two constant in each simulation.

Prob. 6. Thermoelectric system

6-1. (cont'd)

Total 9 simulations



Prob. 6. Thermoelectric system

6-1. (cont'd) which statement is correct?

a. The higher the heat transfer coefficient is, the lower the power output is generated.

b. Optimal power output is only a function of the heat transfer coefficients, and not of the material properties.

c. Optimal power output does not change with varying any of the three individual material properties and the heat transfer coefficients as long as the *ZT* value remains the same.

d. Optimal power output does not change with varying any of the three individual material properties as long as the *ZT* value and the heat transfer coefficients remain the same.

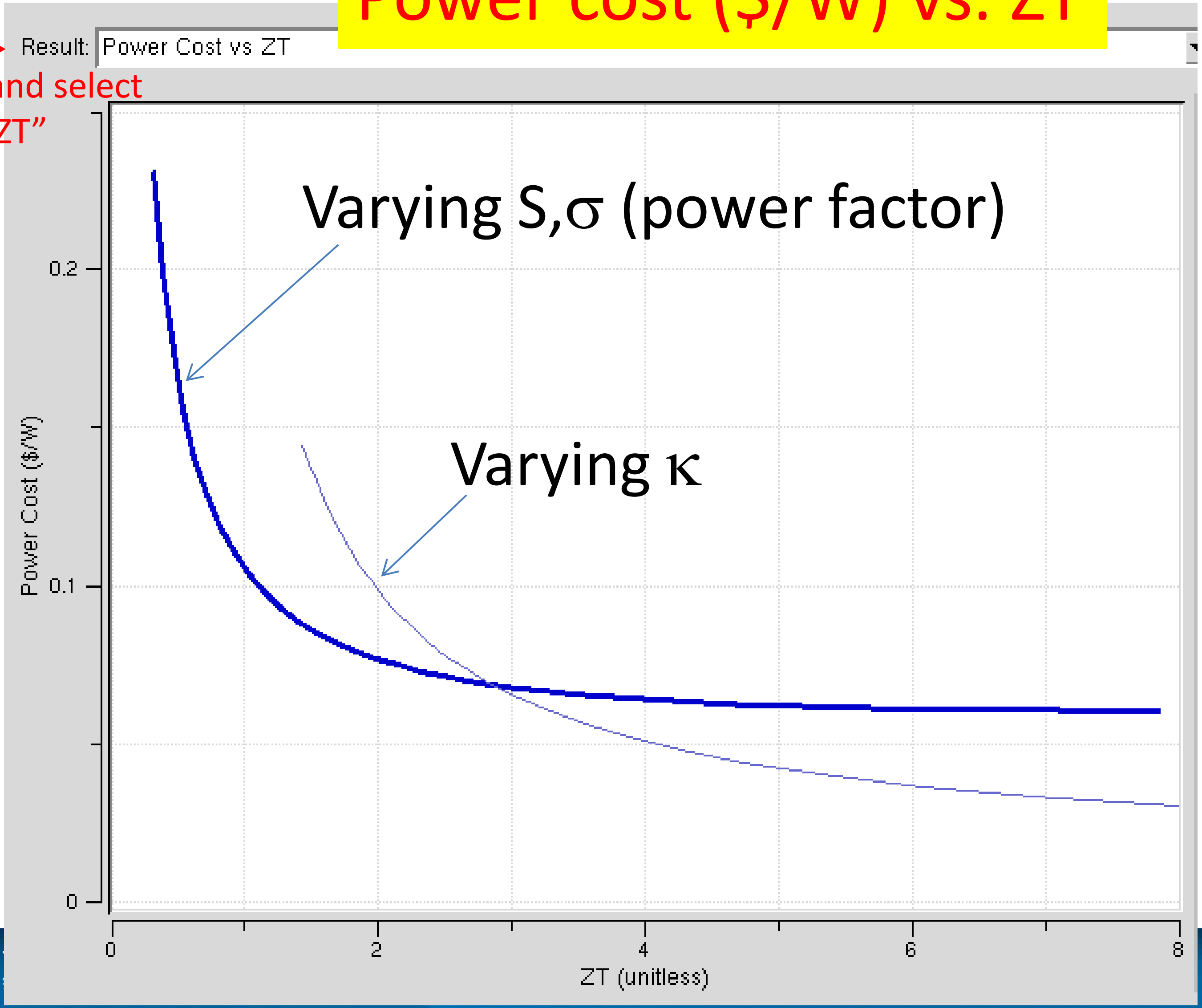
e. None of the above.

Prob. 6. Thermoelectric system

6-2. Cost analysis

Power cost ($/W) vs. ZT

Click the menu and select
“Power cost vs. ZT”



Prob. 6. Thermoelectric system

6-2. (Cost analysis, cont'd) which statement is correct?

a. Materials produce the same power cost if their ZT values are the same.

b. In the high ZT regime (e.g. ZT higher than ~ 3.0), lowering the thermal conductivity is more desired than enhancing the power factor in terms of power cost if ZT remains the same.

c. In the high ZT regime (e.g. ZT higher than ~ 3.0), enhancing the power factor is more desired than lowering the thermal conductivity in terms of power cost if ZT remains the same.

d. Power cost increases with increasing ZT.

e. Power cost decreases with decreasing efficiency.