ECE 659 EXAM V

Monday May 1 2017, 8-10AM

NAME : SOLUTION

CLOSED BOOK

All four questions carry equal weight

Please show all work.

No credit for just writing down the answer, even if correct.

5.1. Consider a system having a very large number of ***degenerate*** levels with energy ε with an electron-electron interaction energy related to the number of electrons N by the relation

$$U(N) = \frac{U_0}{2} N(N-1)$$

At what value of the electrochemical potential μ, will the equilibrium number of electrons change from 100 to 101?

SOLUTION:

States	E - μN
$N = 100$	$100(\epsilon - \mu) + U_0 \times 50 \times 99$
$N = 101$	$101(\epsilon - \mu) + U_0 \times 50 \times 101$

Transition from N=100 to N=101 occurs when

$$(E - \mu N)_{100} = (E - \mu N)_{101}$$

that is , when

$$\epsilon - \mu + 100U_0 = 0$$
$$\rightarrow \mu = \epsilon + 100U_0$$

5.2. Consider a quantum dot with 4 degenerate levels, all with the same energy ε, in equilibrium with a reservoir with electrochemical potential μ and temperature T. If there were no interactions, the average number of electrons at equilibrium would be given by

$$<n> = \frac{4}{1+e^{(\varepsilon-\mu)/kT}}$$

Obtain an expression for the average number of electrons if the electron-electron interaction is so large that no more than one of the 4 levels can be occupied at the same time.

SOLUTION:

There is one zero-electron state with probability

$$P_0 = \frac{1}{Z}$$

and 4 one-electron states each with a probability

$$P_1 = \frac{e^{-x}}{Z}\ , \quad x \equiv \frac{\varepsilon-\mu}{kT}$$

$$<n> = \frac{\sum\limits_n nP_n}{\sum\limits_n P_n} = \frac{4P_1}{P_0+4P_1}$$

$$<n> = \frac{4e^{-x}}{1+4e^{-x}} = \frac{4}{4+e^{x}} = \frac{4}{4+e^{(\varepsilon-\mu)/kT}}$$

5.3. Consider a single quantum dot with ***just one level*** with energy ε. We wish to find the current as a function of the electrochemical potential $\mu_2 = \mu_1 + qV$ (keeping μ_1 fixed) using the Fock space picture by writing

$$\frac{d}{dt}\begin{Bmatrix} P_0 \\ P_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \left[W_1 + W_2\right]\begin{Bmatrix} P_0 \\ P_1 \end{Bmatrix}$$

where the W_1 and W_2 matrices describe the transition rates between the different states due to electron exchange with contacts 1 and 2 respectively which are held in equilibrium with electrochemical potentials μ_1 and μ_2.

(a) Write down the matrices W_1 and W_2 is terms of the Fermi functions in contacts 1, 2.
(b) Obtain the steady-state probabilities P_0 and P_1.
(c) Obtain the steady-state current.

SOLUTION:

(a)

$$W_1 = \gamma_1\begin{bmatrix} -f_1 & 1-f_1 \\ +f_1 & -(1-f_1) \end{bmatrix} \qquad W_2 = \gamma_2\begin{bmatrix} -f_2 & 1-f_2 \\ +f_2 & -(1-f_2) \end{bmatrix}$$

(b)

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} -\gamma_1 f_1 - \gamma_2 f_2 & \gamma_1(1-f_1)+\gamma_2(1-f_2) \\ \gamma_1 f_1 + \gamma_2 f_2 & -\gamma_1(1-f_1)-\gamma_2(1-f_2) \end{bmatrix}\begin{Bmatrix} P_0 \\ P_1 \end{Bmatrix}$$

$$\frac{P_1}{P_0} = \frac{\gamma_1 f_1 + \gamma_2 f_2}{\gamma_1(1-f_1)+\gamma_2(1-f_2)} \;\rightarrow\; P_1 = \frac{\gamma_1 f_1 + \gamma_2 f_2}{\gamma_1 + \gamma_2}, \quad P_0 = 1 - P_1$$

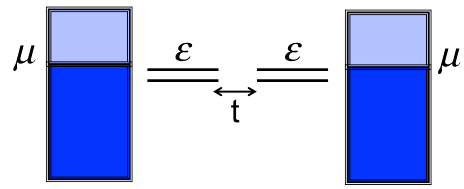
(c)

$$I = q\gamma_1\begin{Bmatrix} 0 & 1 \end{Bmatrix}\begin{bmatrix} -f_1 & 1-f_1 \\ +f_1 & -(1-f_1) \end{bmatrix}\begin{Bmatrix} P_0 \\ P_1 \end{Bmatrix} = q\gamma_1\left(f_1 P_0 - (1-f_1)P_1\right) = \gamma_1(f_1 - P_1)$$

$$= q\gamma_1\left(f_1 - \frac{\gamma_1 f_1 + \gamma_2 f_2}{\gamma_1 + \gamma_2}\right) = q\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}(f_1 - f_2)$$

5.4. Consider two coupled quantum dots each with two spin-degenerate levels, described by the one-electron Hamiltonian,

$$h = \begin{array}{c} \begin{matrix} u_1 & u_2 & d_1 & d_2 \end{matrix} \\ \begin{bmatrix} \varepsilon & t & 0 & 0 \\ t & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & t \\ 0 & 0 & t & \varepsilon \end{bmatrix} \end{array}$$



having an intra-dot interaction energy U and zero inter-dot interaction energy.

We have seen that the Hamiltonian for two-electron states can be written as

$$H_2 = \begin{array}{c} \begin{matrix} u_1d_1 & u_2d_2 & u_1d_2 & u_2d_1 & u_1u_2 & d_1d_2 \end{matrix} \\ \begin{bmatrix} 2\varepsilon+U & 0 & t & t & 0 & 0 \\ 0 & 2\varepsilon+U & t & t & 0 & 0 \\ t & t & 2\varepsilon & 0 & 0 & 0 \\ t & t & 0 & 2\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\varepsilon \end{bmatrix} \end{array}$$

which can be transformed into

$$\begin{array}{c} \begin{matrix} \frac{u_1d_1+u_2d_2}{\sqrt{2}} & \frac{u_1d_1-u_2d_2}{\sqrt{2}} & \frac{u_1d_2+u_2d_1}{\sqrt{2}} & \frac{u_1d_2-u_2d_2}{\sqrt{2}} & u_1u_2 & d_1d_2 \end{matrix} \\ \begin{bmatrix} 2\epsilon+U & 0 & 2t & 0 & 0 & 0 \\ 0 & 2\epsilon+U & 0 & 0 & 0 & 0 \\ 2t & 0 & 2\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\epsilon \end{bmatrix} \end{array}$$

Explain how you would obtain the lowest energy eigenvalue and the corresponding eigenvector making use of the result that the eigenvectors of

$$\vec{\sigma}.\hat{n} \equiv \sigma_x \sin\theta\cos\phi + \sigma_y \sin\theta\sin\phi + \sigma_z \cos\theta$$

corresponding to eigenvalues +1 and -1 can be written as

$$\begin{Bmatrix} c \\ s \end{Bmatrix}, \begin{Bmatrix} -s^* \\ c^* \end{Bmatrix}$$

respectively, where

$$c \equiv \cos(\theta/2)\, e^{-i\varphi/2}, \quad s \equiv \sin(\theta/2)\, e^{+i\varphi/2}$$

SOLUTION:

The lowest energy eigenvalue comes from the 2x2 block

$$\begin{array}{cc} \frac{u_1 d_1 + u_2 d_2}{\sqrt{2}} & \frac{u_1 d_2 + u_2 d_1}{\sqrt{2}} \end{array}$$

$$\begin{bmatrix} 2\epsilon + U & 2t \\ 2t & 2\epsilon \end{bmatrix}$$

which can be written as

$$\left(2\epsilon + \frac{U}{2}\right) I + \frac{U}{2}\sigma_z + 2t\sigma_x$$

$$= \left(2\epsilon + \frac{U}{2}\right) I + B cos\theta\ \sigma_z + B sin\theta\ \sigma_x$$

$$= \left(2\epsilon + \frac{U}{2}\right) I + B\ \vec{\sigma}.\hat{n}$$

$$where\ B = \sqrt{\left(\frac{U}{2}\right)^2 + (2t)^2}$$

$$and\ tan\theta = \frac{4t}{U}$$

The lowest eigenvalue is

$$\left(2\epsilon + \frac{U}{2}\right) - B = \left(2\epsilon + \frac{U}{2}\right) - \sqrt{\left(\frac{U}{2}\right)^2 + (2t)^2}$$

And the corresponding eigenvector is

$$-sin\frac{\theta}{2}\ \frac{u_1 d_1 + u_2 d_2}{\sqrt{2}} + cos\frac{\theta}{2}\ \frac{u_1 d_2 + u_2 d_1}{\sqrt{2}}$$

Have a great summer !!