SPRING 2016 ECE 659 EXAM I

Friday, Jan.29, 2016, FNY B124, 230-320PM

NAME : ____

CLOSED BOOK
1 page of notes provided

All five questions carry equal weight

1.1. Describe how you obtain the relation

$$I \;=\; \frac{q}{h} M \left(\mu^{+} - \mu^{-}\right) \qquad (A)$$

starting from the general expression

$$I \;=\; \frac{q}{h} \int_{-\infty}^{+\infty} dE\, \tilde{M}(E) \left(f^{+}(E) - f^{-}(E)\right) \qquad (B)$$

and obtain an expression for M. Please state your assumptions clearly.

Solution:

Assume $\quad f^{\pm}(E) = \dfrac{1}{1 + \exp\left(\dfrac{E - \mu^{\pm}}{kT}\right)} \quad , \qquad \bar{\mu} = \dfrac{1}{2}\left(\mu^{+} + \mu^{-}\right)$

and $\mu^{+} - \mu^{-} \;\; << \;\; kT$.

$$f^{+}(E) - f^{-}(E) \;\approx\; \left(\frac{\partial f}{\partial \mu}\right)_{\mu = \bar{\mu}} (\mu^{+} - \mu^{-}) \;=\; \left(-\frac{\partial \bar{f}}{\partial E}\right) (\mu^{+} - \mu^{-})$$

Substituting into (B) we obtain

$$I \;=\; \frac{q}{h} \underbrace{\int_{-\infty}^{+\infty} dE\, \tilde{M}(E) \left(-\frac{\partial \bar{f}}{\partial E}\right)}_{\equiv M} \left(\mu^{+} - \mu^{-}\right)$$

which leads to (A) with M defined as shown.

1.2. For a 3D conductor (area: A, Length: L) with an energy-momentum relation

$$E^2 = \quad E_g^2 + v_0^2 p^2$$

(a) Find the functions M(E), D(E) for positive E (E > 0).

(b) Show that the following relation is satisfied:

$$D(E)v(E)\,p(E) \quad = \quad N(E).d$$

Solution:

(a)

$$N(E) = \frac{4\pi}{3} AL \left(\frac{p}{h}\right)^3 = \frac{4\pi}{3h^3} AL \left(\frac{E^2 - E_g^2}{v_0^2}\right)^{3/2} = \frac{4\pi}{3h^3 v_0^3} AL \left(E^2 - E_g^2\right)^{3/2}$$

$$D(E) = \frac{dN(E)}{dE} = \frac{4\pi}{h^3 v_0^3} AL \left(E^2 - E_g^2\right)^{1/2} E$$

$$M(E) = \pi A \left(\frac{p}{h}\right)^2 = \frac{\pi A}{h^2} \left(\frac{E^2 - E_g^2}{v_0^2}\right)$$

(b) From (a),

$$\frac{N(E)}{D(E)} = \quad \frac{E^2 - E_g^2}{3E}$$

Also, $v(E)p(E) = \dfrac{dE}{dp} p = \quad \dfrac{2v_0^2 p^2}{2E} = \quad \dfrac{E^2 - E_g^2}{E}$

Hence,

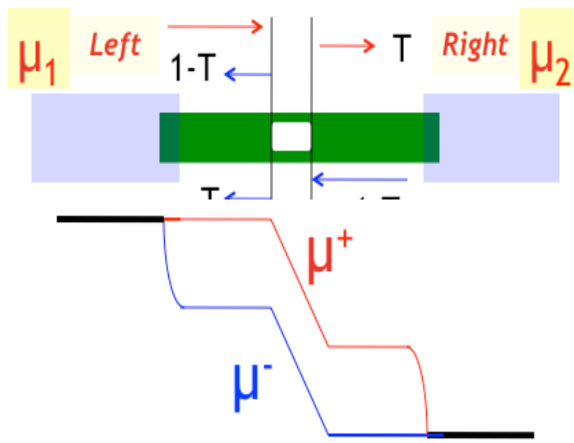
$$D(E)v(E)p(E) = N(E).3$$

The relation is satisfied.

1.3. Consider an otherwise ballistic channel with M modes having a scatterer in the middle where only a fraction T of all the electrons proceed along the original direction, while the rest (1-T) get turned around.

(a) Determine the values of μ^+ and μ^- on either side of the scatterer in terms of μ_1, μ_2 and T and (b) explain why the resistance associated with the voltage drop across the scatterer is given by $R_{scatterer} = \frac{h}{q^2 M}\frac{1-T}{T}$ while the total resistance is given by $R_{total} = \frac{h}{q^2 M T}$

Solution:



(**a**) Since

$$I^{\pm} = (qM/h)\,\mu^{\pm}$$

we can write

$$\mu_2^+ = T\,\mu_1^+ + (1-T)\,\mu_2^-$$

$$\mu_1^- = (1-T)\,\mu_1^+ + T\,\mu_2^-$$

Assume $\mu_1^+ = \mu_1$ and $\mu_2^- = \mu_2$:

$$\mu_2^+ = \mu_2 + T(\mu_1 - \mu_2)$$

$$\mu_1^- = \mu_1 - T(\mu_1 - \mu_2)$$

(b) $I = I^+ - I^- = (qM/h)(\mu_2^+ - \mu_2^-) = (qM/h)(\mu_1^+ - \mu_1^-)$

$= (qMT/h)(\mu_1 - \mu_2)$ Same answer on Left or Right

$$R_{total} \equiv \frac{\mu_1 - \mu_2}{qI} = \frac{h}{q^2 M T}$$

(c) $R_{scatterer} = \frac{1}{2}\frac{(\mu_1^+ + \mu_1^-) - (\mu_2^+ + \mu_2^-)}{qI} = \frac{h}{q^2 M}\frac{1-T}{T}$

1.4. (a) A three terminal conductor is described by

$$I_m = \frac{1}{q}\sum_n G_{mn}\left(\mu_m - \mu_n\right)$$

where $$G = \frac{q^2}{h}\begin{bmatrix} 10 & 6 & 4 \\ 4 & 10 & 6 \\ 6 & 4 & 10 \end{bmatrix}$$

Is this consistent with the requirement of current conservation? Explain

Solution: Current conservation (or Kirchoff's law) requires

$$0 = \sum_m I_m = \frac{1}{q}\sum_{m,n} G_{mn}\left(\mu_m - \mu_n\right) = \frac{1}{q}\sum_{m,n}\left(G_{mn} - G_{nm}\right)\mu_m$$

Since this must be true regardless of the values of μ_m

$$\sum_n\left(G_{mn} - G_{nm}\right) = 0 \quad \rightarrow \quad \sum_n G_{mn} = \sum_n G_{nm}$$

The given G-matrix satisfies this requirement and hence is consistent with current conservation.

(b) A two terminal conductor is described by

$$I_m = \frac{1}{q}\sum_n G_{mn}\left(\mu_m - \mu_n\right)$$

where $$G = \frac{q^2}{h}\begin{bmatrix} 10 & 6 \\ 4 & 10 \end{bmatrix}$$

Is this consistent with current conservation?

Solution: This G-matrix does not satisfy the above requirement and hence is not consistent with current conservation.

1.5. Evaluate the left hand side of the steady-state Boltzmann equation

$$v_z \frac{\partial f_0}{\partial z} + F_z \frac{\partial f_0}{\partial p_z}$$

where f_0 is the equilibrium Fermi function with $E = \varepsilon(p_z) + U(z)$.

$$f_0(E) \equiv \frac{1}{1 + \exp\left(\frac{E - \mu_0}{kT}\right)}$$

Note : $v_z \equiv \frac{d\varepsilon}{dp_z}$**,** $F_z \equiv -\frac{dU}{dz}$

Solution:

$$v_z \frac{\partial f_0}{\partial z} + F_z \frac{\partial f_0}{\partial p_z} \quad = \quad \frac{\partial f_0}{\partial E}\left(v_z \frac{\partial E}{dz} + F_z \frac{\partial E}{dp_z}\right) \quad = \quad \frac{\partial f_0}{\partial E}\left(v_z \frac{dU}{dz} + F_z \frac{d\varepsilon}{dp_z}\right) \quad = \quad 0$$

since $v_z \equiv \frac{d\varepsilon}{dp_z}$**,** $F_z \equiv -\frac{dU}{dz}$