ECE 659 PRACTICE EXAM V

Actual Exam V
Tuesday, May 3, 2016 8-10AM, FRNY B124

CLOSED BOOK

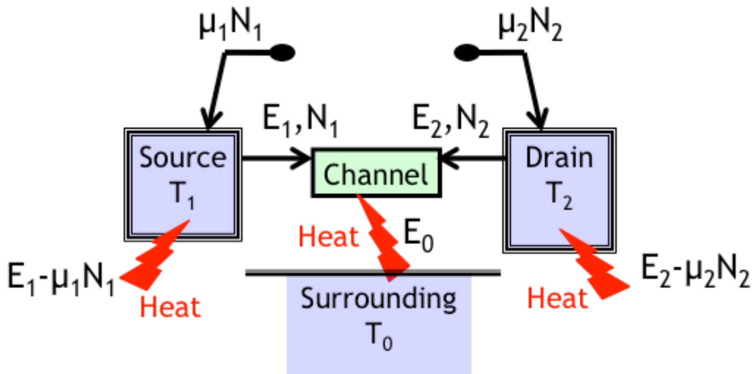
Actual Exam will have five questions.

The following questions have been chosen to stress what I consider the most important concepts / skills that you should be clear on.

5.1. Based on: Introduction to Lecture 16 of Text (LNE)
5.2. Based on: Section 16.1, 16.2 of LNE
5.3. Based on: Section 16.3, 16.4 of LNE
5.4. Based on: Section 16.3, 16.4 of LNE
5.5. Based on: Section 16.5 of LNE
5.6. Based on: Section 24.3 of LNE
5.7. Based on: Sections 23.1-23.2 of LNE, Chapter 3 of QTAT (Reference)
5.8, 5.9. Based on: Sections 23.1-23.2 of LNE, Chapter 3 of QTAT (Reference)
5.10. Singlet and triplet states: Ref. Section 23.3 of LNE

The following topics will NOT be covered on the exam, but may be of interest to you.
A. *Fuel value of information:* Lecture 17, Section 24.4 of LNE
B. *Inelastic scattering:* Ref. Chapter 10 of QTAT

5.1.



Consider a channel exchanging electrons and energies with three reservoirs as shown.

(a) What relationship among the different quantities (N_1, N_2, E_1, E_2, E_0) is required by
- (i) Conservation of number of electrons
- (ii) Conservation of energy
- (iii) Second law of thermodynamics:

(b) Explain why for an elastic resistor our current formula

$$I = \frac{1}{q}\int dE\, G(E)\, \left(f_1(E) - f_2(E)\right)$$

is in compliance with the second law.

(c) Two materials A and B have different density of states D_A and D_B for electrons in a certain energy range. If these materials are in equilibrium with electrons flowing freely between them, will the number of electrons in this energy range in the two materials n_A and n_B be equal? If not, what will be the ratio $\frac{n_A}{n_B}$? Explain.

SOLUTION:

Please see introductory material in Lecture 16 of Text (LNE)

5.2. Two systems are brought into close contact and allowed to exchange energy. Which way will the heat flow, from the one with higher dS/dE to the one with the lower dS/dE or the other way? Explain why, assuming that the total entropy must increase.
(S : Entropy, E: Energy)

SOLUTION:

$$\Delta S = -\left(\frac{dS}{dE}\right)_1 E(1 \to 2) - \left(\frac{dS}{dE}\right)_2 E(2 \to 1)$$

$$E(1 \to 2) = -E(2 \to 1)$$

$$\Delta S = \left[\left(\frac{dS}{dE}\right)_2 - \left(\frac{dS}{dE}\right)_1\right] E(1 \to 2)$$

But $\quad \Delta S \geq 0$

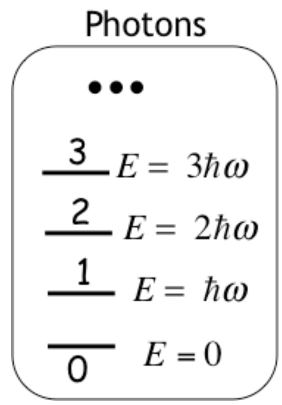
Hence $\quad \left(\frac{dS}{dE}\right)_2 > \left(\frac{dS}{dE}\right)_1 \qquad if \;\; E(1 \to 2) > 0$

That is, heat flows from reservoir with smaller dS/dE to one with larger dS/dE

5.3. Consider photons with energy $\hbar\omega$ having energy levels as shown. Assuming that different states are occupied with probability

$$P_i = \frac{1}{Z} e^{-E_i/kT}$$

obtain an expression for the average number of photons.



SOLUTION:

$$P_n = \frac{1}{Z} e^{-nx} \ , \ x \equiv \frac{\hbar\omega}{kT}$$

$$< n > = \frac{\sum_n n P_n}{\sum_n P_n} = \frac{\sum_n n e^{-nx}}{\sum_n e^{-nx}}$$

$$\sum_n e^{-nx} = 1 + e^{-x} + \cdots = \frac{1}{1 - e^{-x}}$$

Since $\displaystyle \frac{d}{dx}\sum_n e^{-nx} = \frac{d}{dx}\left(\frac{1}{1-e^{-x}}\right) = -\frac{e^{-x}}{(1-e^{-x})^2}$

we have $\displaystyle \sum_n n\, e^{-nx} = \frac{e^{-x}}{(1-e^{-x})^2}$

Hence $\displaystyle < n > = \frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^{x}-1} = \frac{1}{e^{\hbar\omega/kT}-1}$

5.4. Consider a quantum dot with "g" degenerate levels, all with the same energy ε, in equilibrium with a reservoir with electrochemical potential μ and temperature T. If there were no interactions, the average number of electrons at equilibrium would be given by

$$< n > = \frac{g}{1 + e^{(\varepsilon - \mu)/kT}}$$

Assuming that different states are occupied with probability

$$P_i = \frac{1}{Z} e^{-E_i/kT}$$

obtain an expression for the average number of electrons if the electron-electron interaction is so large that no more than one of the "g" levels can be occupied at the same time.

SOLUTION:

There is one zero-electron state with probability

$$P_0 = \frac{1}{Z}$$

and "g" one-electron states each with a probability

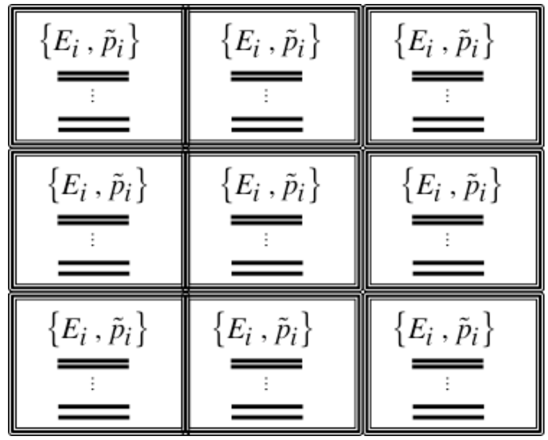
$$P_1 = \frac{e^{-x}}{Z}, \quad x \equiv \frac{\varepsilon - \mu}{kT}$$

$$< n > = \frac{\sum_n n P_n}{\sum_n P_n} = \frac{g P_1}{P_0 + g P_1}$$

$$< n > = \frac{g e^{-x}}{1 + g e^{-x}} = \frac{g}{g + e^{x}} = \frac{g}{g + e^{(\varepsilon - \mu)/kT}}$$

5.5. Suppose the reservoir is composed of many identical subsystems each with energy levels $\tilde{E}_i$ occupied according to probabilities $\tilde{p}_i$. Show that the entropy can be expressed as

$$S = \quad k\, \ell n W(\tilde{p}_i) \quad = \quad -k \sum_i \tilde{p}_i\, \ell n\, \tilde{p}_i$$



SOLUTION:

See Section 16.5.1 of notes.

5.6. What is the entropy per spin (ρ : density matrix)

$$S \;=\; -k\ Trace[\rho\ \ell n\ \rho]$$

corresponding to each of these four collections of spins

a) All pointing along +z
b) All pointing along +x
c) Randomly pointing along ±z
d) Randomly pointing along ±x

(a) $\rho \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$S \;= -k\ Trace\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \ell n 1 & 0 \\ 0 & \ell n 0 \end{bmatrix} \;=\; 0$$

(b) $\rho \;=\; \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Can transform basis so that

$$\rho \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

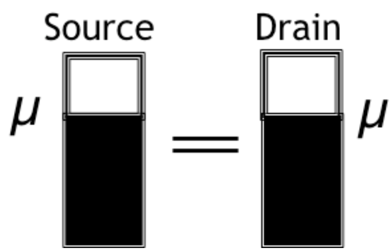
since Trace is invariant under basis transformation S = 0 as in (a).

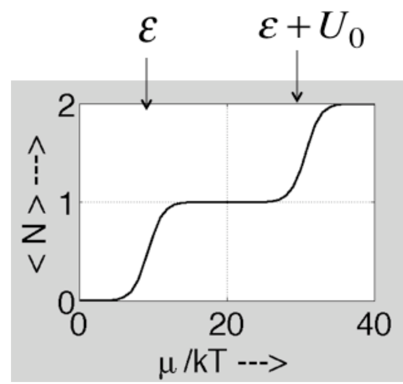
(c), (d) $\rho \;=\; \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$S \;= -k\ Trace\ \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} \ell n 1/2 & 0 \\ 0 & \ell n 1/2 \end{bmatrix} \;=\; k\ \ell n 2$$

5.7. The plot below shows the average number of electrons as a function of the electrochemical potential μ for a single quantum dot with two spin-degenerate levels with energy ε = 10kT and interaction energy U=20kT, in equilibrium with a reservoir with an electrochemical potential μ.

Explain the values of μ at which the transitions in <N> from 0 to 1 and from 1 to 2 occur. Hint: Consider low temperatures, such that $p_i \approx 1$ for state that has the minimum value of (E_i - μN_i) and $p_i \approx 0$ for all other states.





SOLUTION:

States	$E_i - \mu N_i$
00	0
01	$\varepsilon - \mu$
10	$\varepsilon - \mu$
11	$2\varepsilon + U_0 - 2\mu$

Transition from N=0 to N=1 occurs when

$$\varepsilon - \mu_1 = 0 \rightarrow \quad \mu_1 = \varepsilon$$

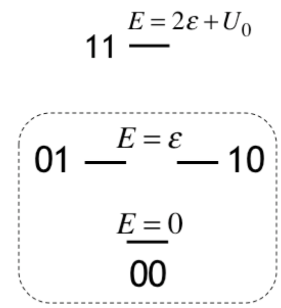
Transition from N=1 to N=2 occurs when

$$\varepsilon - \mu_2 = 2\varepsilon + U_0 - 2\mu_2 \rightarrow \quad \mu_2 = \varepsilon + U_0$$

Probs. 5.8, 5.9 are based on the following description on this page.

The plot below shows the current as a function of the electrochemical potential $\mu_2 = \mu_1 + qV$ (keeping μ_1 fixed) for a single quantum dot with two spin-degenerate levels with energy ε = 10kT and interaction energy U=20kT.

4 many-electron levels



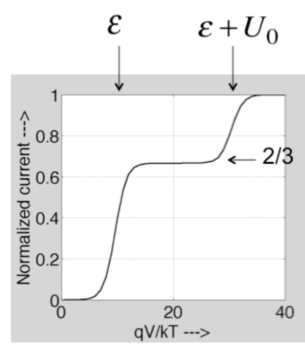
These plots are obtained numerically by first calculating the steady-state occupation probabilities for each of the four many-electron levels using the equation

$$\frac{d}{dt}\begin{Bmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = [W_1 + W_2]\begin{Bmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{Bmatrix} \qquad (1)$$

and then finding the current from either of the relations:

$$I = \{0 \quad 1 \quad 1 \quad 2\}[W_1]\begin{Bmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{Bmatrix}$$

$$= -\{0 \quad 1 \quad 1 \quad 2\}[W_2]\begin{Bmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{Bmatrix} \qquad (2)$$



Here the W_1 and W_2 matrices describing the transition rates between the different states due to electron exchange with contacts 1 and 2 respectively:

$$W_1 = \gamma_1 \begin{bmatrix} -2f_1 & 1-f_1 & 1-f_1 & 0 \\ f_1 & -(1-f_1+f_1^{'}) & 0 & 1-f_1^{'} \\ f_1 & 0 & -(1-f_1+f_1^{'}) & 1-f_1^{'} \\ 0 & f_1^{'} & f_1^{'} & -2(1-f_1^{'}) \end{bmatrix} \qquad (3)$$

$$\text{where} \quad f_1 \equiv \frac{1}{1+e^{(\varepsilon-\mu_1)/kT}}, \qquad f_1^{'} \equiv \frac{1}{1+e^{(\varepsilon+U_0-\mu_1)/kT}} \qquad (4)$$

γ_1 is a constant denoting the rate at which an electron in the dot escapes into contact 1 if a state is available. The matrix W_2 is written similarly, replacing $\gamma_1, f_1, f_1^{'}$ with the corresponding quantities $\gamma_2, f_2, f_2^{'}$. The two rates γ_1 and γ_2 were assumed equal in the simulation.

5.8. The current-voltage characteristics show two current plateaus one approximately 2/3 of the other. To understand the first plateau assume

$f_1 = 1, f_2 = 0, f_1' = 0, f_2' = 0:$

In words, electrons with energy ε are available from contact 1 but not from contact 2 and electrons with energy $\varepsilon + U_0$ are not available from either contact.

(a) What are the probabilities P_{00}, P_{01}, P_{10} and P_{11} ?

(b) What is the current ?

SOLUTION: Assume $\gamma_1 = \gamma_2 = \gamma$, from (3)

$$W_1 = \gamma \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad W_2 = \gamma \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

From (1),

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \gamma \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{Bmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{Bmatrix}$$

Easy to show that

$\mathbf{P_{00} = P_{01} = P_{10} = 1/3 \text{ and } P_{11} = 0}$

satisfies the above equation.

From (2),

$$I = \gamma \{0 \quad 1 \quad 1 \quad 2\} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{Bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \end{Bmatrix}$$

$$= \gamma \{0 \quad 1 \quad 1 \quad 2\} \begin{Bmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 0 \end{Bmatrix} = \frac{2}{3}\gamma$$

5.9. To understand the second plateau assume

$f_1 = 1,\ f_2 = 0,\ f_1' = 1,\ f_2' = 0,$

In words, electrons with energy ε as well as energy $\varepsilon + U_0$ are available from contact 1, but not from contact 2.

(a) What are the probabilities P_{00}, P_{01}, P_{10} and P_{11} ?
(b) What is the current ?

SOLUTION: Assume $\gamma_1 = \gamma_2 = \gamma$, from (3)

$$W_1 = \gamma \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad W_2 = \gamma \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

From (1),

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \gamma \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix} \begin{Bmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{Bmatrix}$$

Easy to show that

$\mathbf{P_{00} = P_{01} = P_{10} = P_{11} = 1/4}$

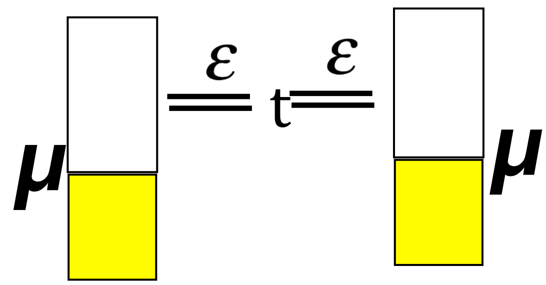
satisfies the above equation.

From (2),

$$I = \gamma\{0 \quad 1 \quad 1 \quad 2\} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{Bmatrix}$$

$$= \gamma\{0 \quad 1 \quad 1 \quad 2\} \begin{Bmatrix} -1/2 \\ 0 \\ 0 \\ 1/2 \end{Bmatrix} = \gamma$$

5.10. Consider two coupled quantum dots each with two spin-degenerate levels, described by the one-electron Hamiltonian,



$$h = \begin{array}{c} \begin{matrix} u_1 & u_2 & d_1 & d_2 \end{matrix} \\ \begin{bmatrix} \varepsilon & t & 0 & 0 \\ t & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & t \\ 0 & 0 & t & \varepsilon \end{bmatrix} \end{array}$$

having an intra-dot interaction energy U and zero inter-dot interaction energy.

Useful result: Eigenvalues of $\begin{bmatrix} a & 0 & t & t \\ 0 & a & t & t \\ t & t & b & 0 \\ t & t & 0 & b \end{bmatrix}$ *are* **a, b and**

$$\frac{(a+b) \pm \sqrt{(a-b)^2 + 16t^2}}{2}$$

(a) The eigenvalues of which matrix will give you the energies of the ***one-electron*** states? What is the lowest energy, assuming t is a negative number?

(b) The eigenvalues of which matrix will give you the energies of the ***two-electron*** states? What is the lowest energy?

(c) At what values of μ does the total number of electrons change from N = 0 to 1 to 2?

(a)

$$H_1 = \begin{array}{c} \begin{matrix} u_1 & u_2 & d_1 & d_2 \end{matrix} \\ \begin{bmatrix} \varepsilon & t & 0 & 0 \\ t & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & t \\ 0 & 0 & t & \varepsilon \end{bmatrix} \end{array}$$

$$\textit{Lowest energy} = \varepsilon + t$$

assuming t is negative.

(b)

$$H_2 = \begin{array}{c} \begin{array}{cccccc} u_1d_1 & u_2d_2 & u_1d_2 & u_2d_1 & u_1u_2 & d_1d_2 \end{array} \\ \begin{bmatrix} 2\varepsilon+U & 0 & t & t & 0 & 0 \\ 0 & 2\varepsilon+U & t & t & 0 & 0 \\ t & t & 2\varepsilon & 0 & 0 & 0 \\ t & t & 0 & 2\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\varepsilon \end{bmatrix} \end{array}$$

$$Lowest\ Energy = \frac{2\varepsilon+(2\varepsilon+U)-\sqrt{U^2+16t^2}}{2}$$

$$= \quad 2\varepsilon+\frac{U}{2}-\sqrt{\left(\frac{U}{2}\right)^2+4t^2} \quad \equiv \quad 2\varepsilon-\Delta$$

$$where \quad \Delta \equiv \sqrt{\left(\frac{U}{2}\right)^2+4t^2}-\frac{U}{2}$$

$$(c) \quad 0 = \quad \varepsilon+t-\mu_1 \quad \rightarrow \quad \mu_1 = \varepsilon+t$$

$$\varepsilon+t-\mu_2 = \quad 2\varepsilon-\Delta-2\mu_2 \quad \rightarrow \quad \mu_2 = \varepsilon-t-\Delta$$