ECE 659 EXAM V

Tuesday May 6, 2014, FRNY B124 8-10AM

NAME : SOLUTION

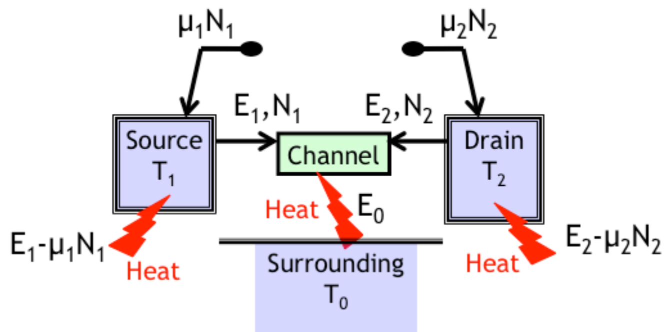
CLOSED BOOK

All five questions carry equal weight

Please show all work.

No credit for just writing down the answer, even if correct.

5.1.



Consider a channel exchanging electrons and energies with three reservoirs as shown.

(a) What relationship among the different quantities (N_1, N_2, E_1, E_2, E_0) is required by
(i) Conservation of number of electrons
(ii) Conservation of energy
(iii) Second law of thermodynamics:

(b) Is our current formula for an elastic resistor

$$I = \frac{1}{q}\int dE\, G(E)\, \left(f_1(E) - f_2(E)\right)$$

in compliance with the second law ? Explain

SOLUTION:

Please see introductory material (before Section 16.1)
in Lecture 16 of Text (LNE)

5.2. Consider a system having three ***degenerate*** levels with energy ε having an electron-electron interaction energy related to the number of electrons N by the relation

$$U(N) = \frac{U_0}{2} N(N-1)$$

At what values of the electrochemical potential μ, will the equilibrium number of electrons change from 0 to 1, from 1 to 2 and from 2 to 3?

SOLUTION:

States	$E_i - \mu N_i$
000	0
001,010,100	$\varepsilon - \mu$
011,101,110	$2\varepsilon + U_0 - 2\mu$
111	$3\varepsilon + 3U_0 - 3\mu$

Transition from N=0 to N=1 occurs when

$$\varepsilon - \mu_1 = 0 \rightarrow \quad \mu_1 = \varepsilon$$

Transition from N=1 to N=2 occurs when

$$\varepsilon - \mu_2 = 2\varepsilon + U_0 - 2\mu_2 \rightarrow \quad \mu_2 = \varepsilon + U_0$$

Transition from N=2 to N=3 occurs when

$$3\varepsilon + 3U_0 - 3\mu_3 = 2\varepsilon + U_0 - 2\mu_3 \rightarrow \quad \mu_3 = \varepsilon + 2U_0$$

5.3. Consider a single quantum dot with ***just one level*** with energy ε. We wish to find the current as a function of the electrochemical potential $\mu_2 = \mu_1 + qV$ (keeping μ_1 fixed) using the Fock space picture by writing

$$\frac{d}{dt}\begin{Bmatrix} P_0 \\ P_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = [W_1 + W_2]\begin{Bmatrix} P_0 \\ P_1 \end{Bmatrix}$$

where the W_1 and W_2 matrices describe the transition rates between the different states due to electron exchange with contacts 1 and 2 respectively which are held in equilibrium with electrochemical potentials μ_1 and μ_2.

(a) Write down the matrices W_1 and W_2 is terms of the Fermi functions in contacts 1, 2.
(b) Obtain the steady-state probabilities P_0 and P_1.
(c) Obtain the steady-state current.

SOLUTION:

(a)

$$W_1 = \gamma_1 \begin{bmatrix} -f_1 & 1-f_1 \\ +f_1 & -(1-f_1) \end{bmatrix} \qquad W_2 = \gamma_2 \begin{bmatrix} -f_2 & 1-f_2 \\ +f_2 & -(1-f_2) \end{bmatrix}$$

(b)

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} -\gamma_1 f_1 - \gamma_2 f_2 & \gamma_1(1-f_1) + \gamma_2(1-f_2) \\ \gamma_1 f_1 + \gamma_2 f_2 & -\gamma_1(1-f_1) - \gamma_2(1-f_2) \end{bmatrix}\begin{Bmatrix} P_0 \\ P_1 \end{Bmatrix}$$

$$\frac{P_1}{P_0} = \frac{\gamma_1 f_1 + \gamma_2 f_2}{\gamma_1(1-f_1) + \gamma_2(1-f_2)} \rightarrow P_1 = \frac{\gamma_1 f_1 + \gamma_2 f_2}{\gamma_1 + \gamma_2}, \quad P_0 = 1 - P_1$$

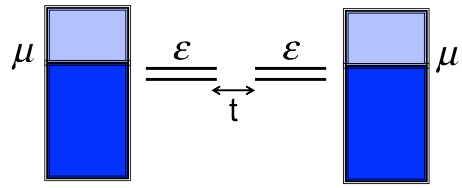
(c)

$$I = q\gamma_1 \{0 \quad 1\}\begin{bmatrix} -f_1 & 1-f_1 \\ +f_1 & -(1-f_1) \end{bmatrix}\begin{Bmatrix} P_0 \\ P_1 \end{Bmatrix} = q\gamma_1 \left(f_1 P_0 - (1-f_1)P_1\right) = \gamma_1(f_1 - P_1)$$

$$= q\gamma_1\left(f_1 - \frac{\gamma_1 f_1 + \gamma_2 f_2}{\gamma_1 + \gamma_2}\right) = q\frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2}(f_1 - f_2)$$

5.4. Consider two coupled quantum dots each with two spin-degenerate levels, described by the one-electron Hamiltonian,

$$h = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & d_1 & d_2 \end{array} \\ \begin{bmatrix} \varepsilon & t & 0 & 0 \\ t & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & t \\ 0 & 0 & t & \varepsilon \end{bmatrix} \end{array}$$



having an intra-dot interaction energy U and zero inter-dot interaction energy. Write down the matrices whose eigenvalues will give you the energies of the

(a) ***Zero-electron*** states

(b) ***One-electron*** states

(c) ***Two-electron*** states

(d) ***Three-electron*** states

(a) ***Four-electron*** states

SOLUTION:

(a) Zero-electron states

$$H_0 = \begin{array}{c} 0 \\ \begin{bmatrix} 0 \end{bmatrix} \end{array}$$

(b) One-electron states

$$H_1 = h = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & d_1 & d_2 \end{array} \\ \begin{bmatrix} \varepsilon & t & 0 & 0 \\ t & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & t \\ 0 & 0 & t & \varepsilon \end{bmatrix} \end{array}$$

(c) Two-electron states

$$H_2 = \begin{array}{c} \begin{array}{cccccc} u_1d_1 & u_2d_2 & u_1d_2 & u_2d_1 & u_1u_2 & d_1d_2 \end{array} \\ \begin{bmatrix} 2\varepsilon+U & 0 & t & t & 0 & 0 \\ 0 & 2\varepsilon+U & t & t & 0 & 0 \\ t & t & 2\varepsilon & 0 & 0 & 0 \\ t & t & 0 & 2\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\varepsilon \end{bmatrix} \end{array}$$

(d) Three-electron states

$$H_3 = \begin{array}{cccc} u_2d_1d_2 & u_1d_1d_2 & u_1u_2d_2 & u_1u_2d_1 \\ \end{array}$$
$$H_3 = \begin{bmatrix} 3\varepsilon+U & t & 0 & 0 \\ t & 3\varepsilon+U & 0 & 0 \\ 0 & 0 & 3\varepsilon+U & t \\ 0 & 0 & t & 3\varepsilon+U \end{bmatrix}$$

(e) Four-electron states

$$\begin{array}{c} u_1u_2d_1d_2 \\ \left[2\varepsilon+2U\right] \end{array}$$

5.5. What is the entropy per spin (ρ : density matrix)

$$S \quad = \quad -k \;\; Trace[\rho \; \ell n \; \rho]$$

corresponding to a collection of spins,

(a) 50% of which point along +x and 50% along -x ?
(b) 75% of which point along +x and 25% along -x ?
(c) 100% of which point along +x?

Note: Wavefunction for (a) spin along +x : $\frac{1}{\sqrt{2}}\begin{Bmatrix}1\\1\end{Bmatrix}$, (b) for spin along -x : $\frac{1}{\sqrt{2}}\begin{Bmatrix}1\\-1\end{Bmatrix}$

SOLUTION:

$$\text{(a)}\;\; \rho \quad = \quad 0.5*\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \;\; +0.5*\frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \;\; \begin{bmatrix}1/2 & 0\\0 & 1/2\end{bmatrix}$$

$$S \quad = -k \;\; Trace \;\; \begin{bmatrix}1/2 & 0\\0 & 1/2\end{bmatrix}\begin{bmatrix}\ell n 1/2 & 0\\0 & \ell n 1/2\end{bmatrix} \quad = \quad k \; \ell n 2$$

$$\text{(b)}\;\; \rho \quad = \quad 0.75*\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \;\; +0.25*\frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \;\; \begin{bmatrix}1/2 & 1/4\\1/4 & 1/2\end{bmatrix}$$

Diagonalizing

$$\rho \quad = \quad \begin{bmatrix}3/4 & 0\\0 & 1/4\end{bmatrix}$$

Hence

$$S \quad = -k \;\; Trace \;\; \begin{bmatrix}3/4 & 0\\0 & 1/4\end{bmatrix}\begin{bmatrix}\ell n 3/4 & 0\\0 & \ell n 1/4\end{bmatrix} \quad = \quad -k\left(\frac{3}{4}\ell n\frac{3}{4}+\frac{1}{4}\ell n\frac{1}{4}\right)$$

$$\text{(c)}\;\; \rho \quad = \quad \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

Diagonalizing

$$\rho \quad = \quad \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

Hence

$$S \quad = -k \;\; Trace \;\; \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\ell n 1 & 0\\0 & \ell n 0\end{bmatrix} \quad = \quad 0$$