SPRING 2015 ECE 659 EXAM I

Friday, Jan.30, 2015, EE 117, 230-320PM

NAME : ____

CLOSED BOOK

1 page of notes provided

All five questions carry equal weight.

Please show all work.
No credit for just writing down the answer, even if correct.

1.1. Describe how you obtain the relation

$$I = \frac{q}{h} M_0 \left(\mu^+ - \mu^-\right) \qquad (A)$$

starting from the general expression

$$I = \frac{q}{h} \int_{-\infty}^{+\infty} dE\, M(E) \left(f^+(E) - f^-(E)\right) \qquad (B)$$

and relate M_0 to M(E). Please state your assumptions clearly.

Solution:

Assume $$f^{\pm}(E) = \frac{1}{1 + \exp\left(\dfrac{E - \mu^{\pm}}{kT}\right)} \; , \qquad \bar{\mu} = \frac{1}{2}\left(\mu^+ + \mu^-\right)$$

and $\mu^+ - \mu^- \ll kT$.

$$f^+(E) - f^-(E) \approx \left(\frac{\partial f}{\partial \mu}\right)_{\mu=\mu_0} (\mu^+ - \mu^-) = \left(-\frac{\partial f_0}{\partial E}\right) (\mu^+ - \mu^-)$$

Substituting into (B) we obtain

$$I = \frac{q}{h} \underbrace{\int_{-\infty}^{+\infty} dE\, M(E) \left(-\frac{\partial f_0}{\partial E}\right)}_{\equiv M_0} \left(\mu^+ - \mu^-\right)$$

which leads to (A) and expresses M_0 in terms of M(E).

1.2. Consider electrons obeying the relativistic energy-momentum relation

$$E^2 = m^2c^4 + p^2c^2$$

1.

(a) Find the velocity as a function of the momentum.

(b) Find the momentum as a function of the velocity.

Solution:

(a)

$$2E\frac{dE}{dp} = 2c^2p \to \quad v \equiv \frac{dE}{dp} = \frac{c^2p}{E} = \frac{c^2p}{\sqrt{m^2c^4 + p^2c^2}}$$

$$v = \frac{c\,p}{\sqrt{m^2c^2 + p^2}}$$

(b)

$$v^2(m^2c^2 + p^2) = c^2p^2 \to \quad p^2 = \frac{m^2v^2c^2}{c^2 - v^2} \quad \to \quad p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

1.3. For a 3D conductor (area: A, Length: L) with an energy-momentum relation

$$E^2 = E_g^2 + v_0^2 p^2$$

(a) find the functions M(E), D(E) for positive E (E > 0).

(b) Is the relation

$$D(E)v(E)p(E) = N(E).d$$

satisfied?

Solution:

(a)

$$N(E) = \frac{4\pi}{3} AL\left(\frac{p}{h}\right)^3 = \frac{4\pi}{3h^3} AL\left(\frac{E^2 - E_g^2}{v_0^2}\right)^{3/2} = \frac{4\pi}{3h^3 v_0^3} AL\left(E^2 - E_g^2\right)^{3/2}$$

$$D(E) = \frac{dN(E)}{dE} = \frac{4\pi}{h^3 v_0^3} AL\left(E^2 - E_g^2\right)^{1/2} E$$

$$M(E) = \pi A\left(\frac{p}{h}\right)^2 = \frac{\pi A}{h^2}\left(\frac{E^2 - E_g^2}{v_0^2}\right)$$

(b) From (a), $$\frac{N(E)}{D(E)} = \frac{E^2 - E_g^2}{3E}$$

Also, $$v(E)p(E) = \frac{dE}{dp} p = \frac{2v_0^2 p^2}{2E} = \frac{E^2 - E_g^2}{E}$$

Hence,

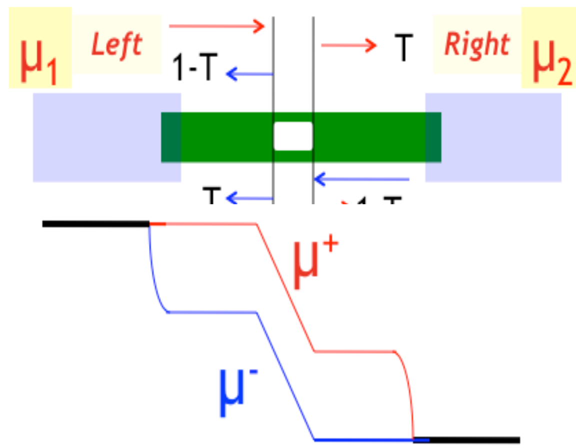
$$D(E)v(E)p(E) = N(E).3$$

The relation is satisfied.

1.4. Consider an otherwise ballistic channel with M modes having a scatterer in the middle where only a fraction T of all the electrons proceed along the original direction, while the rest (1-T) get turned around.

Determine the values of μ^+ and μ^- on either side of the scatterer in terms of μ_1, μ_2 and T

Solution:



Since

$$I^{\pm} = (qM/h)\,\mu^{\pm}$$

we can write

$$\mu_2^+ \;=\; T\,\mu_1^+ \;+\; (1-T)\,\mu_2^-$$

$$\mu_1^- \;=\; (1-T)\,\mu_1^+ \;+\; T\,\mu_2^-$$

Assume $\mu_1^+ = \mu_1$ and $\mu_2^- = \mu_2$:

$$\mu_2^+ \;=\; \mu_2 + \; T(\mu_1 - \mu_2)$$

$$\mu_1^- \;=\; \mu_1 - \; T(\mu_1 - \mu_2)$$

1.5. Evaluate the left hand side of the Boltzmann equation

$$\frac{\partial f_0}{\partial t} + v_z \frac{\partial f_0}{\partial z} + F_z \frac{\partial f_0}{\partial p_z}$$

where f_0 is the equilibrium Fermi function with $E = \varepsilon(p_z) + U(z)$.

$$f_0(E) \equiv \frac{1}{1+\exp\left(\dfrac{E-\mu_0}{kT}\right)}$$

Note : $v_z \equiv \dfrac{d\varepsilon}{dp_z}$, $F_z \equiv -\dfrac{dU}{dz}$

Solution:

$$v_z \frac{\partial f_0}{\partial z} + F_z \frac{\partial f_0}{\partial p_z} \quad = \quad \frac{\partial f_0}{\partial E}\left(v_z \frac{\partial E}{dz} + F_z \frac{\partial E}{dp_z}\right) \quad = \quad \frac{\partial f_0}{\partial E}\left(v_z \frac{dU}{dz} + F_z \frac{d\varepsilon}{dp_z}\right) \quad = \quad 0$$

since $v_z \equiv \dfrac{d\varepsilon}{dp_z}$, $F_z \equiv -\dfrac{dU}{dz}$